A new look at the KLMN theorem

Damir Kinzebulatov (Laval) and Yuli A. Semenov (Toronto)

May 25, 2018

The KLMN Theorem (Kato-Lions-Lax-Milgram-Nelson)

Part I: The existing results

Schrödinger operator $-\Delta - V$
Kolmogorov backward operator $-\Delta + b \cdot \nabla$

Schrödinger operator

$$-\Delta - V, \quad V \geq 0$$

on $\mathbb{R}^d$, $d \geq 3$

KLMN

If $V \in L^1_{\mathrm{loc}}$ is "small" relative to $-\Delta$, i.e. there exists $\delta < 1$ such that[1]

$$\langle V^{\frac{1}{2}}\varphi, V^{\frac{1}{2}}\varphi\rangle \leq \delta\langle \nabla\varphi, \nabla\varphi\rangle + c\langle\varphi,\varphi\rangle, \quad \varphi \in C_c^\infty \qquad (\mathbf{F}_\delta)$$

then $-\Delta - V$ has a self-adjoint operator realization in L^2

[1] $\langle u\rangle := \int_{\mathbb{R}^d} u(x)dx, \quad \langle u, v\rangle := \langle u\bar{v}\rangle.$

Schrödinger operator

$$-\Delta - V, \quad V \geq 0$$

on $\mathbb{R}^d$, $d \geq 3$

KLMN

If $V \in L^1_{\rm loc}$ is "small" relative to $-\Delta$, i.e. there exists $\delta < 1$ such that[1]

$$\langle V^{\frac{1}{2}}\varphi, V^{\frac{1}{2}}\varphi\rangle \leq \delta\langle \nabla\varphi, \nabla\varphi\rangle + c\langle\varphi,\varphi\rangle, \quad \varphi \in C_c^\infty \qquad (\mathbf{F}_\delta)$$

then $-\Delta - V$ has a self-adjoint operator realization in L^2

$\mathbf{F}_\delta$: the potential energy $\langle V\varphi, \varphi\rangle$ is dominated by the kinetic energy $\langle\nabla\varphi, \nabla\varphi\rangle$

($\Rightarrow$ the ground state energy of the quantum mechanical system is finite)

[1] $\langle u\rangle := \int_{\mathbb{R}^d} u(x)dx, \quad \langle u, v\rangle := \langle u\bar{v}\rangle.$

Schrödinger operator

$$-\Delta - V, \quad V \geq 0$$

on $\mathbb{R}^d$, $d \geq 3$

KLMN

If $V \in L^1_{\rm loc}$ is "small" relative to $-\Delta$, i.e. there exists $\delta < 1$ such that

$$\langle V^{\frac{1}{2}}\varphi, V^{\frac{1}{2}}\varphi\rangle \leq \delta\langle\nabla\varphi, \nabla\varphi\rangle + c\langle\varphi,\varphi\rangle, \quad \varphi \in C_c^\infty \qquad (\mathbf{F}_\delta)$$

then $-\Delta - V$ has a self-adjoint operator realization in L^2

$$V \in L^{\frac{d}{2}} \quad \Rightarrow \quad V \in \mathbf{F}_\delta \text{ with arbitrarily small } \delta$$

$$V(x) := k|x|^{-2} \quad \Rightarrow^{2} \quad V \in \mathbf{F}_\delta \text{ with } \delta = \frac{4k}{(d-2)^2}$$

[2] Hardy's inequality

[2] Hardy's inequality

Schrödinger operator

$$-\Delta - V, \quad V \geq 0$$

on $\mathbb{R}^d$, $d \geq 3$

KLMN

If $V \in L^1_{\mathrm{loc}}$ is "small" relative to $-\Delta$, i.e. there exists $\delta < 1$ such that

$$\langle V^{\frac{1}{2}}\varphi, V^{\frac{1}{2}}\varphi\rangle \leq \delta\langle\nabla\varphi, \nabla\varphi\rangle + c\langle\varphi,\varphi\rangle, \quad \varphi \in C_c^\infty \qquad (\mathbf{F}_\delta)$$

then $-\Delta - V$ has a self-adjoint operator realization in L^2

$$V \in L^{\frac{d}{2}} \quad \Rightarrow \quad V \in \mathbf{F}_\delta \text{ with arbitrarily small } \delta$$

$$V(x) := k|x|^{-2} \quad \Rightarrow^{2} \quad V \in \mathbf{F}_\delta \text{ with } \delta = \frac{4k}{(d-2)^2}$$

Note: for any $\varepsilon > 0$ there exist $V \in \mathbf{F}_\delta$ such that $V \notin L^{1+\varepsilon}_{\mathrm{loc}}$

[2] Hardy's inequality

[2] Hardy's inequality

Schrödinger operator

$$-\Delta - V, \quad V \geq 0$$

on $\mathbb{R}^d$, $d \geq 3$

KLMN

If $V \in L^1_{\rm loc}$ is "small" relative to $-\Delta$, i.e. there exists $\delta < 1$ such that

$$\langle V^{\frac{1}{2}}\varphi, V^{\frac{1}{2}}\varphi\rangle \leq \delta\langle \nabla\varphi, \nabla\varphi\rangle + c\langle\varphi,\varphi\rangle, \quad \varphi \in C_c^\infty \qquad (\mathbf{F}_\delta)$$

then $-\Delta - V$ has a self-adjoint operator realization in L^2

We need to **give up the arithmetic sum**, i.e. $-\Delta - V$ of the domain

$$D(-\Delta) \cap D(V) \equiv W^{2,2} \cap \{u \in L^2 : Vu \in L^2\}$$

since the latter may be **not dense** in L^2 (e.g. if $V \notin L^2$)...

Schrödinger operator

$$-\Delta - V, \quad V \geq 0$$

on $\mathbb{R}^d$, $d \geq 3$

KLMN

If $V \in L^1_{\mathrm{loc}}$ is "small" relative to $-\Delta$, i.e. there exists $\delta < 1$ such that

$$\langle V^{\frac{1}{2}}\varphi, V^{\frac{1}{2}}\varphi\rangle \leq \delta\langle \nabla\varphi, \nabla\varphi\rangle + c\langle\varphi,\varphi\rangle, \quad \varphi \in C_c^\infty \qquad (\mathbf{F}_\delta)$$

then $-\Delta - V$ has a self-adjoint operator realization in L^2

A proof of the KLMN via quadratic form

$$\mathfrak{t}[u] := \langle \nabla u, \nabla u\rangle + \langle Vu, u\rangle, \quad D(\mathfrak{t}) = W^{1,2}$$

(closed, symmetric, semi-bounded from below)

Schrödinger operator

$$-\Delta - V, \quad V \geq 0$$

on $\mathbb{R}^d$, $d \geq 3$

KLMN

If $V \in L^1_{\rm loc}$ is "small" relative to $-\Delta$, i.e. there exists $\delta < 1$ such that

$$\langle V^{\frac{1}{2}}\varphi, V^{\frac{1}{2}}\varphi\rangle \leq \delta\langle\nabla\varphi,\nabla\varphi\rangle + c\langle\varphi,\varphi\rangle, \quad \varphi \in C_c^\infty \qquad (\mathbf{F}_\delta)$$

then $-\Delta - V$ has a self-adjoint operator realization in L^2

A proof of the KLMN via quadratic form

$$\mathfrak{t}[u] := \langle\nabla u,\nabla u\rangle + \langle Vu,u\rangle, \quad D(\mathfrak{t}) = W^{1,2}$$

(closed, symmetric, semi-bounded from below) $\Rightarrow$ $\mathfrak{t}$ is the quadratic form of a self-adjoint operator Λ on L^2:

$$D(\Lambda) \subset D(\mathfrak{t}), \qquad \langle\Lambda u,v\rangle = \mathfrak{t}[u,v] \quad (u \in D(\Lambda), v \in D(\mathfrak{t}))$$

□

P. Stollmann, J. Voigt: an example of $V \notin L^q_{\rm loc}$, $q > 0$, such that the quadratic forms method nevertheless works

Another approach:

E. Nelson gave a different proof of the KLMN Theorem which is based on the construction of J. L. Lions (the standard triple of Hilbert spaces. . .)

(Will return to this in a moment)

The next important operator (subject of this talk): the Kolmogorov backward operator

$$-\Delta + b \cdot \nabla, \quad b : \mathbb{R}^d \to \mathbb{R}^d$$

(in Diffusion Processes ...)

The next important operator (subject of this talk): the Kolmogorov backward operator

$$-\Delta + b \cdot \nabla, \quad b : \mathbb{R}^d \to \mathbb{R}^d$$

(in Diffusion Processes ...)

Next 5 min: Will consider a somewhat more general operator,

$$-\Delta + b \cdot \nabla + \nabla \cdot \tilde{b} + V$$

and state the general KLMN Theorem

Starting object: a densely defined, closed, sectorial sesquilinear form $\mathfrak{t}$ on L^2

Sectorial: The numerical range of the quadratic form $\mathfrak{t}[u] := \mathfrak{t}[u, u]$

$$\Theta(\mathfrak{t}) := \{\mathfrak{t}[u] : \|u\| = 1\} \subset \mathbb{C}$$

is a subset of the sector

$$\{\zeta \in \mathbb{C} : |\arg(\zeta - \gamma)| \leq \theta\}, \quad 0 \leq \theta < \frac{\pi}{2}, \quad \gamma \in \mathbb{R}$$

(e.g. if $\mathfrak{t}[u]$ is semi-bounded from below: $\mathfrak{t}[u] \geq \gamma\|u\|$, $\gamma \in \mathbb{R}$)

Starting object: a densely defined, closed, sectorial sesquilinear form $\mathfrak{t}$ on L^2

Sectorial: The numerical range of the quadratic form $\mathfrak{t}[u] := \mathfrak{t}[u, u]$

$$\Theta(\mathfrak{t}) := \{\mathfrak{t}[u] : \|u\| = 1\} \subset \mathbb{C}$$

is a subset of the sector

$$\{\zeta \in \mathbb{C} : |\arg(\zeta - \gamma)| \leq \theta\}, \quad 0 \leq \theta < \frac{\pi}{2}, \quad \gamma \in \mathbb{R}$$

(e.g. if $\mathfrak{t}[u]$ is semi-bounded from below: $\mathfrak{t}[u] \geq \gamma\|u\|$, $\gamma \in \mathbb{R}$)

Closed: For every $\{u_n\} \subset D(\mathfrak{t})$

$$\text{If} \quad u_n \to u, \quad \mathfrak{t}[u_n - u_m] \to 0$$

$$\Rightarrow \qquad u \in D(\mathfrak{t}), \quad \mathfrak{t}[u_n - u] \to 0$$

KLMN (T. Kato)

Let $\mathfrak{t}[u,v]$ be a densely defined, closed, sectorial sesquilinear form on L^2

Then there exists a m-accretive operator[3] Λ on L^2 such that

(i) $D(\Lambda) \subset D(\mathfrak{t})$ and $\mathfrak{t}[u,v] = \langle \Lambda u, v \rangle$ $(u \in D(\Lambda),\ v \in D(\mathfrak{t}))$

(ii) If $\mathfrak{t} \leftrightarrow \Lambda$, then[4] $\mathfrak{t}^* \leftrightarrow \Lambda^*$

[3] Λ is uniquely determined by (i)

[4] $\mathfrak{t}^*[u,v] := \overline{\mathfrak{t}[v,u]}$, $D(\mathfrak{t}^*) = D(\mathfrak{t})$

KLMN (T. Kato)

Let $\mathfrak{t}[u,v]$ be a densely defined, closed, sectorial sesquilinear form on L^2

Then there exists a m-accretive operator[3] Λ on L^2 such that

(i) $D(\Lambda) \subset D(\mathfrak{t})$ and $\mathfrak{t}[u,v] = \langle \Lambda u, v \rangle$ $(u \in D(\Lambda),\ v \in D(\mathfrak{t}))$

(ii) If $\mathfrak{t} \leftrightarrow \Lambda$, then[4] $\mathfrak{t}^* \leftrightarrow \Lambda^*$

"Λ is m-accretive": Λ is closed,

$$\text{Re}\langle \Lambda u, u \rangle \geq \gamma \|u\|, \quad \gamma \in \mathbb{R}, \quad u \in D(\Lambda)$$

and $\|(\zeta + \Lambda)^{-1}\|_{L^2 \to L^2} \leq (\text{Re}\zeta)^{-1}$, $\text{Re}\zeta > \gamma$

($\Rightarrow \Lambda$ is the generator of a contraction C_0 semigroup on L^2)

[3] Λ is uniquely determined by (i)

[4] $\mathfrak{t}^*[u,v] := \overline{\mathfrak{t}[v,u]}$, $D(\mathfrak{t}^*) = D(\mathfrak{t})$

Example 1: The form associated to the operator $-\Delta + b \cdot \nabla + \nabla \cdot \tilde{b} - V$,

$$\mathfrak{t}[u,v] := \langle \nabla u, \nabla v \rangle + \langle \nabla u, bv \rangle - \langle \tilde{b} u, \nabla v \rangle + \langle V u, v \rangle, \quad D(\mathfrak{t}) = W^{1,2}$$

satisfies conditions of the KLMN Theorem provided that $|b|^2 \in \mathbf{F}_{\delta_1}$, $|\tilde{b}|^2 \in \mathbf{F}_{\delta_2}$, $V \in \mathbf{F}_\delta$, with $\delta_1 + \delta_2 + \delta < 1$

In particular:

Example 2: The form associated to the operator $-\Delta - V$, $V \geq 0$,

$$\mathfrak{t}[u,v] := \langle \nabla u, \nabla v \rangle - \langle V u, v \rangle, \quad D(\mathfrak{t}) = W^{1,2}$$

satisfies conditions of the KLMN Theorem provided that $V \in \mathbf{F}_\delta$, $\delta < 1$

Example 3: The form associated to the operator $-\Delta + b \cdot \nabla$, $b : \mathbb{R}^d \to \mathbb{R}^d$,

$$\mathfrak{t}[u,v] := \langle \nabla u, \nabla v \rangle + \langle \nabla u, bv \rangle, \quad D(\mathfrak{t}) = W^{1,2}$$

satisfies conditions of the KLMN Theorem provided that $|b|^2 \in \mathbf{F}_\delta$, $\delta < 1$

Example 1: The form associated to the operator $-\Delta + b \cdot \nabla + \nabla \cdot \tilde{b} - V$,

$$\mathfrak{t}[u,v] := \langle \nabla u, \nabla v \rangle + \langle \nabla u, bv \rangle - \langle \tilde{b}u, \nabla v \rangle + \langle Vu, v \rangle, \quad D(\mathfrak{t}) = W^{1,2}$$

satisfies conditions of the KLMN Theorem provided that $|b|^2 \in \mathbf{F}_{\delta_1}$, $|\tilde{b}|^2 \in \mathbf{F}_{\delta_2}$, $V \in \mathbf{F}_\delta$, with $\delta_1 + \delta_2 + \delta < 1$

In particular:

Example 2: The form associated to the operator $-\Delta - V$, $V \geq 0$,

$$\mathfrak{t}[u,v] := \langle \nabla u, \nabla v \rangle - \langle Vu, v \rangle, \quad D(\mathfrak{t}) = W^{1,2}$$

satisfies conditions of the KLMN Theorem provided that $V \in \mathbf{F}_\delta$, $\delta < 1$

Example 3: The form associated to the operator $-\Delta + b \cdot \nabla$, $b : \mathbb{R}^d \to \mathbb{R}^d$,

$$\mathfrak{t}[u,v] := \langle \nabla u, \nabla v \rangle + \langle \nabla u, bv \rangle, \quad D(\mathfrak{t}) = W^{1,2}$$

satisfies conditions of the KLMN Theorem provided that $|b|^2 \in \mathbf{F}_\delta$, $\delta < 1$

Part II: A new look at the KLMN Theorem

Kolmogorov backward operator $-\Delta + b \cdot \nabla$, $b : \mathbb{R}^d \to \mathbb{R}^d$

("$|b|^2 \in \mathbf{F}_\delta$" misses the point)

Kato-Lax-Milgram approach (i.e. quadratic forms) requires $|b|^2 \in \mathbf{F}_\delta$, $\delta < 1$:

$$\langle |b|^2\varphi,\varphi\rangle \leq \delta\langle\nabla\varphi,\nabla\varphi\rangle + c\langle\varphi,\varphi\rangle, \quad \varphi\in C_c^\infty \qquad (\mathbf{F}_\delta)$$

This talk[5]: significant gain in the admissible singularities of b

$$\langle |b|\varphi,\varphi\rangle \leq \delta\langle(\lambda-\Delta)^{\frac{1}{4}}\varphi,(\lambda-\Delta)^{\frac{1}{4}}\varphi\rangle, \quad \varphi\in C_c^\infty$$

(write $|b| \in \mathbf{F}_\delta^{1/2}$)

$\mathbf{F}_\delta \subsetneq \mathbf{F}_\delta^{1/2}$

$\mathbf{F}_\delta^{1/2}$ allows to handle hypersurface singularities of b (precisely: this class contains the Kato class $\mathbf{K}_\delta^{d+1}$)

[5] D. Kinzebulatov, Yu.A. Semenov "On the theory of the Kolmogorov operator in the spaces L^p and C_∞. I" arXiv:1709.08598 (2017)

KLMN Theorem for $-\Delta + b \cdot \nabla$

If $|b| \in \mathbf{F}_\delta^{1/2}$, $\delta < 1$, then $-\Delta + b \cdot \nabla$ has an operator realization Λ on L^2, the generator of a quasi bounded holomorphic semigroup

(+ representation of the resolvent, $L^p \to L^q$ smoothing estimates ...)[6]

Proof: Not clear how to apply quadratic forms

We develop an approach based on the old ideas of J.L. Lions and E. Hille.

[6] D. Kinzebulatov, Yu.A. Semenov "On the theory of the Kolmogorov operator in the spaces L^p and C_∞. I" arXiv:1709.08598 (2017)

First, an outline of the **old approach** based on the **standard triple of Hilbert spaces**.

$\mathcal{H} = L^2$
$\mathcal{H}_+ = W^{1,2}$ (with norm $\|f\|_+^2 := \lambda \|f\|_2^2 + \|(-\Delta)^{\frac{1}{2}} f\|_2^2$)
$\mathcal{H}_- = \mathcal{H}_+^*$

$$\mathcal{H}_+ \subset \mathcal{H} \cong \mathcal{H}^* \subset \mathcal{H}_-$$

is the standard triple of Hilbert spaces with respect to $\langle \ , \ \rangle_{\mathcal{H}}$

...the standard triple of Hilbert spaces

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$$

$\hat{A} \equiv$ the extension of $-\Delta$ by continuity to the operator from $\mathcal{H}_+$ to $\mathcal{H}_-$,

$$|\langle f, (\zeta + \hat{A})f \rangle| \geq \|f\|_+^2, \quad f \in \mathcal{H}_+, \quad \text{Re}\zeta > \lambda$$

Thus, $\zeta + \hat{A} \in \mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$ is a bijection ...

Consider now the perturbation term $\hat{B} \equiv b \cdot \nabla : \mathcal{H}_+ \to \mathcal{H}_-$

If $|b|^2 \in \mathbf{F}_\delta$, $\delta < 1$, then $\hat{B} \in \mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$ and

$$|\langle f, (\zeta + \hat{A} + \hat{B})f\rangle| \geq (1 - \sqrt{\delta})\langle f, (\mu + \hat{A})f\rangle, \qquad \mu = \frac{|2\zeta - \lambda\sqrt{\delta}|}{2(1 - \sqrt{\delta})} > 0$$

whenever Re$\zeta > \frac{\lambda\sqrt{\delta}}{2}$.

Then $\hat{\Lambda} := \hat{A} + \hat{B} \in \mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$ is a bijection. This is the **principal object** in the J. Lions approach[7] (cf. quadratic forms in Kato-Lax-Milgram)

[7] We can have $A \equiv -\nabla \cdot a \cdot \nabla$ in place of $-\Delta$

Consider now the perturbation term $\hat{B} \equiv b \cdot \nabla : \mathcal{H}_+ \to \mathcal{H}_-$

If $|b|^2 \in \mathbf{F}_\delta$, $\delta < 1$, then $\hat{B} \in \mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$ and

$$|\langle f, (\zeta + \hat{A} + \hat{B}) f \rangle| \geq (1 - \sqrt{\delta}) \langle f, (\mu + \hat{A}) f \rangle, \qquad \mu = \frac{|2\zeta - \lambda\sqrt{\delta}|}{2(1 - \sqrt{\delta})} > 0$$

whenever $\mathrm{Re}\zeta > \frac{\lambda\sqrt{\delta}}{2}$.

Then $\hat{\Lambda} := \hat{A} + \hat{B} \in \mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$ is a bijection. This is the **principal object** in the J. Lions approach[7] (cf. quadratic forms in Kato-Lax-Milgram)

We return to $\mathcal{H}$ using E. Hille's theory of pseudo-resolvents ...

[7] We can have $A \equiv -\nabla \cdot a \cdot \nabla$ in place of $-\Delta$

The (new) Hille-Lions approach for $b \in \mathbf{F}_\delta^{1/2}$

Let $\mathcal{H}_0 := L^2$

$A := \lambda - \Delta$ of domain $D(A) = W^{2,2}$

$$\mathcal{H}_\alpha := \big(D(A^\alpha), \langle f, g\rangle_{\mathcal{H}_\alpha} = \langle A^\alpha f, A^\alpha g\rangle\big) \quad (\alpha \geq 0)$$

Consider the following **quintuplet** of Hilbert spaces

$$\mathcal{H}_1 \subset \mathcal{H}_{\frac{3}{4}} \subset \mathcal{H}_{\frac{1}{4}} \subset \mathcal{H}_0 \subset \mathcal{H}_{-\frac{1}{4}}.$$

Then $\mathcal{H}_l \to \mathcal{H}_{l+\frac{1}{4}}$, $l = -\frac{1}{4}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ are bijections

By $\langle f, g\rangle_{\frac{1}{4}}$, $f \in \mathcal{H}_{-\frac{1}{4}}$, $g \in \mathcal{H}_{\frac{3}{4}}$ we denote the pairing between $\mathcal{H}_{-\frac{1}{4}}$ and $\mathcal{H}_{\frac{3}{4}}$, then

$$\langle f, g\rangle_{\frac{1}{4}} = \langle f, g\rangle_{\mathcal{H}_{\frac{1}{4}}} \text{ whenever } f \in \mathcal{H}_{\frac{1}{4}}$$

By $\hat{A}$ we denote the extension by continuity of $A - \lambda$ to the operator from $\mathcal{H}_{\frac{3}{4}}$ into $\mathcal{H}_{-\frac{1}{4}}$. Then

$$|\langle(\zeta + \hat{A})f, f\rangle_{\frac{1}{4}}| \geq \|f\|^2_{\mathcal{H}_{\frac{3}{4}}} \qquad (f \in \mathcal{H}_{\frac{3}{4}},\ \text{Re}\zeta \geq \lambda),$$

and so $\zeta + \hat{A}$ is a bijection; $\|\zeta + \hat{A}\|_{\mathcal{H}_{\frac{3}{4}} \to \mathcal{H}_{-\frac{1}{4}}} \geq 1$. Clearly

$$D(A) = \{f \in \mathcal{H}_{\frac{3}{4}} \mid \hat{A}f \in \mathcal{H}_0\} \text{ and } A^{-1} = (\lambda + \hat{A})^{-1} \upharpoonright \mathcal{H}_0$$

By $|b| \in \mathbf{F}_\delta^{1/2}$, the operator $\hat{B} := b \cdot \nabla : \mathcal{H}_{\frac{3}{4}} \to \mathcal{H}_{-\frac{1}{4}}$ is bounded:

$$b^{\frac{1}{2}} \cdot \nabla : \mathcal{H}_{\frac{3}{4}} \to \mathcal{H}_0, \; |b|^{\frac{1}{2}} : \mathcal{H}_0 \to \mathcal{H}_{-\frac{1}{4}},$$

so

$$\hat{B} \in \mathcal{B}(\mathcal{H}_{\frac{3}{4}}, \mathcal{H}_{-\frac{1}{4}}) \text{ with } \|\hat{B}\|_{\mathcal{H}_{\frac{3}{4}} \to \mathcal{H}_{-\frac{1}{4}}} \leq \delta.$$

By $|b| \in \mathbf{F}_\delta^{1/2}$, the operator $\hat{B} := b\cdot\nabla : \mathcal{H}_{\frac{3}{4}} \to \mathcal{H}_{-\frac{1}{4}}$ is bounded:

$$b^{\frac{1}{2}}\cdot\nabla : \mathcal{H}_{\frac{3}{4}} \to \mathcal{H}_0,\ |b|^{\frac{1}{2}} : \mathcal{H}_0 \to \mathcal{H}_{-\frac{1}{4}},$$

so

$$\hat{B} \in \mathcal{B}(\mathcal{H}_{\frac{3}{4}}, \mathcal{H}_{-\frac{1}{4}}) \text{ with } \|\hat{B}\|_{\mathcal{H}_{\frac{3}{4}}\to\mathcal{H}_{-\frac{1}{4}}} \leq \delta.$$

Thus $|\langle(\zeta + \hat{A} + \hat{B})f, f\rangle_{\frac{1}{4}}| \geq (1-\delta)\|f\|^2_{\mathcal{H}_{\frac{3}{4}}}$, and so

$$\zeta + \hat{\Lambda} := \zeta + \hat{A} + \hat{B} \in \mathcal{B}(\mathcal{H}_{\frac{3}{4}}, \mathcal{H}_{-\frac{1}{4}})$$

is a bijection

By $|b| \in \mathbf{F}_\delta^{1/2}$, the operator $\hat{B} := b \cdot \nabla : \mathcal{H}_{\frac{3}{4}} \to \mathcal{H}_{-\frac{1}{4}}$ is bounded:

$$b^{\frac{1}{2}} \cdot \nabla : \mathcal{H}_{\frac{3}{4}} \to \mathcal{H}_0, \ |b|^{\frac{1}{2}} : \mathcal{H}_0 \to \mathcal{H}_{-\frac{1}{4}},$$

so

$$\hat{B} \in \mathcal{B}(\mathcal{H}_{\frac{3}{4}}, \mathcal{H}_{-\frac{1}{4}}) \text{ with } \|\hat{B}\|_{\mathcal{H}_{\frac{3}{4}} \to \mathcal{H}_{-\frac{1}{4}}} \leq \delta.$$

Thus $|\langle(\zeta + \hat{A} + \hat{B})f, f\rangle_{\frac{1}{4}}| \geq (1-\delta)\|f\|^2_{\mathcal{H}_{\frac{3}{4}}}$, and so

$$\zeta + \hat{\Lambda} := \zeta + \hat{A} + \hat{B} \in \mathcal{B}(\mathcal{H}_{\frac{3}{4}}, \mathcal{H}_{-\frac{1}{4}})$$

is a bijection

Now, the **problem** is how to return to $\mathcal{H}_0$

Set $\hat{R}_\zeta := (\zeta + \hat{\Lambda})^{-1}$

We have

$$\hat{R}_\zeta = \hat{R}_\eta + (\eta - \zeta)\hat{R}_\zeta \hat{R}_\eta \quad (\zeta, \eta \in \mathcal{O}) \tag{p_1}$$

and

$$R_\zeta := \hat{R}_\zeta \restriction \mathcal{H}_0 = (\zeta - \Delta)^{-\frac{3}{4}}(1 + H^* S)^{-1}(\zeta - \Delta)^{-\frac{1}{4}}. \tag{p_2}$$

where $H := |b|^{\frac{1}{2}}(\zeta - \Delta)^{-\frac{1}{4}}$, $S := b^{\frac{1}{2}} \cdot \nabla(\zeta - \Delta)^{-\frac{3}{4}}$, $\|H^* S\|_{2 \to 2} \leq \delta$

By (p_1) that R_ζ is a pseudo-resolvent, by (p_2) its null-set is $\{0\}$. Therefore, R_ζ is the resolvent of some closed operator Λ in $\mathcal{H}_0$,

$$\Lambda = R_\zeta^{-1} - \zeta, \quad D(\Lambda) = R(R_\zeta)$$

(E. Hille).

So, $D(\Lambda) = \hat{R}_\lambda \mathcal{H}_0$, $\Lambda f := \hat{\Lambda} f$, $f \in D(\Lambda)$

Now, $\mathcal{H}_0$ is dense in $\mathcal{H}_{-\frac{1}{4}} \Rightarrow \hat{R}_\lambda \mathcal{H}_0$ is dense in $\mathcal{H}_{-\frac{3}{4}} \Rightarrow \hat{R}_\lambda \mathcal{H}_0$ is dense in $\mathcal{H}_0$
$\Rightarrow \Lambda$ is a densely defined closed operator

Finally, since

$$\begin{aligned}\|H^*Sf\|_{\mathcal{H}_0} &\leq \|H\|_{\mathcal{H}_0 \to \mathcal{H}_0} \||b|^{\frac{1}{2}} J_\lambda |\nabla J_\zeta^2 f|\|_{\mathcal{H}_0} \\ &\leq \delta \|\nabla J_\zeta^2 f|\|_{\mathcal{H}_0} \leq \delta \|f\|_{\mathcal{H}_0},\ f \in \mathcal{H}_0,\end{aligned}$$

it follows from (p_2) that

$$\|R_\zeta\|_{\mathcal{H}_0 \to \mathcal{H}_0} \leq (1-\delta)^{-1} |\zeta|^{-1} \qquad (p_3)$$

i.e. $-\Lambda$ is the generator of a quasi bounded holomorphic semigroup on L^2 □

[8] D. Kinzebulatov, Yu.A. Semenov "On the theory of the Kolmogorov operator in the spaces L^p and C_∞. I" arXiv:1709.08598 (2017)

Finally, since

$$\begin{aligned}\|H^* S f\|_{\mathcal{H}_0} &\leq \|H\|_{\mathcal{H}_0 \to \mathcal{H}_0} \||b|^{\frac{1}{2}} J_\lambda |\nabla J_\zeta^2 f|\|_{\mathcal{H}_0} \\ &\leq \delta \|\nabla J_\zeta^2 f|\|_{\mathcal{H}_0} \leq \delta \|f\|_{\mathcal{H}_0}, \; f \in \mathcal{H}_0,\end{aligned}$$

it follows from (p_2) that

$$\|R_\zeta\|_{\mathcal{H}_0 \to \mathcal{H}_0} \leq (1-\delta)^{-1} |\zeta|^{-1} \qquad (p_3)$$

i.e. $-\Lambda$ is the generator of a quasi bounded holomorphic semigroup on L^2 □

Remark: L^2-theory (provided by the KLMN Theorem) is crucial for getting into L^p and C_∞ (Diffusion Processes etc)[8]

[8] D. Kinzebulatov, Yu.A. Semenov "On the theory of the Kolmogorov operator in the spaces L^p and C_∞. I" arXiv:1709.08598 (2017)

L^2 theory of $-\Delta + b \cdot \nabla$

$|b| \in \mathbf{F}_\delta^{1/2}$, $\delta < 1$

Hille-Trotter approach

Set

$$\Theta(\zeta, b) := (\zeta - \Delta)^{-\frac{3}{4}} \big(1 + P_\zeta(b)\big)^{-1} (\zeta - \Delta)^{-\frac{1}{4}}, \quad \zeta \in \mathcal{O} := \{z : \text{Re} z \geq \lambda\},$$

where $P_\zeta(b) := (\zeta - \Delta)^{-\frac{1}{4}} |b|^{\frac{1}{2}} b^{\frac{1}{2}} \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}}$, $b^{\frac{1}{2}} := |b|^{-\frac{1}{2}} b$

Since $|b| \in \mathbf{F}_\delta^{1/2}$,

$$\|P_\zeta\|_{L^2 \to L^2} \leq \|(\zeta - \Delta)^{-\frac{1}{4}} |b|^{\frac{1}{2}}\|_{L^2 \to L^2} \|b^{\frac{1}{2}} \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}}\|_{L^2 \to L^2} \leq \delta,$$

so $\Theta(\zeta, b) \in \mathcal{B}(L^2)$

Set

$$\Theta(\zeta, b) := (\zeta - \Delta)^{-\frac{3}{4}} \big(1 + P_\zeta(b)\big)^{-1} (\zeta - \Delta)^{-\frac{1}{4}}, \quad \zeta \in \mathcal{O} := \{z : \mathrm{Re} z \geq \lambda\},$$

where $P_\zeta(b) := (\zeta - \Delta)^{-\frac{1}{4}} |b|^{\frac{1}{2}} b^{\frac{1}{2}} \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}}$, $b^{\frac{1}{2}} := |b|^{-\frac{1}{2}} b$

Since $|b| \in \mathbf{F}_\delta^{1/2}$,

$$\|P_\zeta\|_{L^2 \to L^2} \leq \|(\zeta - \Delta)^{-\frac{1}{4}} |b|^{\frac{1}{2}}\|_{L^2 \to L^2} \|b^{\frac{1}{2}} \cdot \nabla (\zeta - \Delta)^{-\frac{3}{4}}\|_{L^2 \to L^2} \leq \delta,$$

so $\Theta(\zeta, b) \in \mathcal{B}(L^2)$

The operator-valued function $\Theta(\zeta, b)$ is the **principal object** in this approach ($\equiv$ a candidate for the resolvent of $-\Delta + b \cdot \nabla$)

Let

$$b_n := \mathbf{1}_n b, \quad n = 1, 2, \dots,$$

where $\mathbf{1}_n$ is the indicator of $\{x \in \mathbb{R}^d \mid |x| \leq n, |b(x)| \leq n\}$

$\|b_n \cdot \nabla(\eta - \Delta)^{-1}\|_{L^2 \to L^2} < 1$, $\eta > n^2$, so by Miyadera's Perturbation Theorem, $-\tilde{\Lambda}(b_n) := \Delta - b_n \cdot \nabla$ of domain $W^{2,2}$ generates a C_0 semigroup in L^2

By iterating the resolvent identity,

$$\Theta(\eta, b_n) = (\eta + \tilde{\Lambda}(b_n))^{-1}, \quad \eta > n^2 \vee \lambda \qquad (p_1)$$

Direct calculations show:

$$\Theta(\zeta, b_n) - \Theta(\eta, b_n) = (\eta - \zeta)\Theta(\zeta, b_n)\Theta(\eta, b_n), \quad \zeta, \eta \in \mathcal{O}, \qquad (p_2)$$

so $\Theta(\zeta, b_n)$ is a pseudo-resolvent on $\mathcal{O}$

By (p_2), the range of $\Theta(\zeta, b_n)$ equals to the range of $\Theta(\eta, b_n)$ and hence by (p_1) is dense in L^2

Therefore,

$$\Theta(\zeta, b_n) = (\zeta + \tilde{\Lambda}(b_n))^{-1}, \quad \zeta \in \mathcal{O}, \quad n \geq 1$$

By the Dominated Convergence Theorem,

$$\Theta(\zeta, b_n) \xrightarrow{s} \Theta(\zeta, b)$$

By the definition of $\Theta(\zeta, b)$,

$$\|\Theta(\zeta, b_n)\|_{L^2\to L^2} \leq c|\zeta|^{-1}, \quad \zeta \in \mathcal{O},$$

and[9]

$$\mu\Theta(\mu, b_n) \xrightarrow{s} 1 \quad \text{as } \mu \to \infty$$

So, by the Trotter Approximation Theorem $\Theta(\zeta, b)$ is indeed the resolvent of an operator $-\Lambda(b)$ that generates a quasi bounded holomorphic semigroup on L^2

$\square$

[9] $\Theta(\zeta, b_n) = (\zeta-\Delta)^{-1} - (\zeta-\Delta)^{-\frac{3}{4}} H^*(1+SH^*)^{-1} S(\zeta-\Delta)^{-\frac{1}{4}}$, where $H := |b|^{\frac{1}{2}}(\zeta-\Delta)^{-\frac{1}{4}}$, $b^{\frac{1}{2}} := |b|^{-\frac{1}{2}} b$, $S := b^{\frac{1}{2}} \cdot \nabla(\zeta-\Delta)^{-\frac{3}{4}}$

By the Dominated Convergence Theorem,

$$\Theta(\zeta, b_n) \xrightarrow{s} \Theta(\zeta, b)$$

By the definition of $\Theta(\zeta, b)$,

$$\|\Theta(\zeta, b_n)\|_{L^2 \to L^2} \leq c|\zeta|^{-1}, \quad \zeta \in \mathcal{O},$$

and[9]

$$\mu\Theta(\mu, b_n) \xrightarrow{s} 1 \quad \text{as } \mu \to \infty$$

So, by the Trotter Approximation Theorem $\Theta(\zeta, b)$ is indeed the resolvent of an operator $-\Lambda(b)$ that generates a quasi bounded holomorphic semigroup on L^2

$\square$

Remark: Both "holomorphic" and the approximation are crucial for this approach

[9] $\Theta(\zeta, b_n) = (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{3}{4}} H^*(1 + SH^*)^{-1} S(\zeta - \Delta)^{-\frac{1}{4}}$, where $H := |b|^{\frac{1}{2}}(\zeta - \Delta)^{-\frac{1}{4}}$, $b^{\frac{1}{2}} := |b|^{-\frac{1}{2}} b$, $S := b^{\frac{1}{2}} \cdot \nabla(\zeta - \Delta)^{-\frac{3}{4}}$

To summarize:

$|b|^2 \in \mathbf{F}_\delta$ $\begin{cases} \text{Kato-Lax-Milgram approach} \\ \text{Hille-Lions approach } (\textbf{new}) \end{cases}$

…and a significant gain in the admissible singularities of b:

$|b| \in \mathbf{F}_\delta^{1/2}$ $\begin{cases} \text{A variant of the Hille-Lions approach } (\textbf{new}) \\ \text{Hille-Trotter approach } (\textbf{new}) \Rightarrow L^p,\ C_\infty \text{ theory, SDEs} \end{cases}$

Let Y be a (complex) Banach space. A pseudo-resolvent R_ζ is a function defined on a subset $\mathcal{O}$ of the complex ζ-plane with values in $\mathcal{B}(Y)$ such that

$$R_\zeta - R_\eta = (\eta - \zeta) R_\zeta R_\eta, \quad \zeta, \eta \in \mathcal{O}.$$

Clearly, R_ζ have common null-set.

Theorem (E. Hille)

If the null-set of R_ζ is $\{0\}$, then R_ζ is the resolvent of a closed linear operator A, the range of R_ζ coincides with $D(A)$, and $A = R_\zeta^{-1} - \zeta$.

Proof: Put $A := R_\zeta^{-1} - \zeta$. Since R_ζ is closed, so is R_ζ^{-1} and A. A straightforward calculation shows that $(\zeta + A) R_\zeta f = f$, $f \in Y$, and $R_\zeta(\zeta + A) g = g$, $g \in D(A)$, as needed.

Theorem (E. Hille)

If there exists a sequence of numbers $\{\mu_k\} \subset \mathcal{O}$ such that $\lim_k |\mu_k| = \infty$ and $\sup_k \|\mu_k R_{\mu_k}\|_{Y \to Y} < \infty$, then the set $\{y \in Y : \lim_k \mu_k R_{\mu_k} y = y\}$ is contained in the closure of the range of R_ζ.

Proof: Indeed, let $\lim_k \mu_k R_{\mu_k} y = y$. That is, for every $\varepsilon > 0$, there exists k such that $\|y - \mu_k R_{\mu_k} y\| < \varepsilon$, so y belongs to the closure of the range of R_ζ.

Consider a sequence of C_0 semigroups e^{-tA_k} on a (complex) Banach space Y.

Theorem (H.F. Trotter)

Let

$$\sup_k \|(\mu + A_k)^{-1}\|_{Y \to Y} \leq \mu^{-1}, \quad \mu > \omega,$$

or

$$\sup_k \|(z + A_k)^{-1}\|_{Y \to Y} \leq C|z|^{-1}, \quad \operatorname{Re} z > \omega,$$

and let $s\text{-}\lim_{\mu \to \infty} \mu(\mu + A_k)^{-1} = 1$ uniformly in k. Let $s\text{-}\lim_k (\zeta + A_k)^{-1}$ exist for some ζ with $\operatorname{Re} \zeta > \omega$. Then there is a C_0 semigroup e^{-tA} such that

$$(z + A_k)^{-1} \xrightarrow{s} (z + A)^{-1} \quad \text{for every } \operatorname{Re} z > \omega,$$

and

$$e^{-tA_k} \xrightarrow{s} e^{-tA}$$

uniformly in any finite interval of $t \geq 0$.

THEOREM (Hille's Pertrubation Theorem)

Let e^{-tA} be a symmetric Markov semigroup, K a linear operator in L^r for some $r \in]1,\infty[$. If for some $\lambda > 0$

$$\|K(\lambda + A_r)^{-1}\|_{r\to r} < \frac{1}{2},$$

then $-\Lambda_r := -A_r - K$ of domain $D(A_r)$ is the generator of a quasi bounded holomorphic semigroup on L^r.

THEOREM (Miyadera's Pertrubation Theorem)

Let e^{-tA} be a symmetric Markov semigroup, K a linear operator in L^r for some $r \in [1,\infty[$. If for some $\lambda > 0$

$$\|K(\lambda + A_r)^{-1}\|_{r\to r} < 1,$$

then $-\Lambda_r := -A_r - K$ of domain $D(A_r)$ is the generator of a quasi bounded C_0 semigroup on L^r.

$$\|f\|_{p,\infty} := \left(\sup_{t>0}\ t^p \mu\{|f(x)| > t\}\right)^{\frac{1}{p}}$$