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Geometric function theory on special domains

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The (very recent) prehistory

In the end of 1990's N. J. Young and his collaborators (among others J. Agler, D. Ogle, J. A. Ball, F. B. Yeh) inspired by problems from μ-synthesis started to investigate **the symmetrized bidisc** (and then its higher dimensional counterpart **symmetrized polydisc**).

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History - continued

Around 2007 another domain having its origin in μ-synthesis (called **tetrablock**) started to be investigated (by N. J. Young & Co.: A. A. Abouhajar, M. C. White). And astonishingly it shared many properties of the symmetrized bidisc although there is no clear reason why it happens so.

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The symmetrized polydisc - definition

Let $\pi^n : \mathbb{C}^n \mapsto \mathbb{C}^n$ be *the symmetrization mapping* given by the formula

$$\pi_k^n(\lambda) := \sum_{1 \le j_1 < \ldots < j_k \le n} \lambda_{j_1} \cdot \ldots \cdot \lambda_{j_k}, \ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n. \quad (1)$$

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$\mathbb{G}_n$ is the set of n-tuples $(s_1, \ldots, s_n) = \pi^n(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that the polynomial

$$\zeta^n - s_1\zeta^{n-1} + s_2\zeta^{n-2} + \ldots + (-1)^n s_n = (\zeta - \lambda_1) \cdot \ldots \cdot (\zeta - \lambda_n) \ (2)$$

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Since the time of introducing the symmetrized polydisc turned out to be important in the geometric function theory. Let us recall the properties which make it important in this theory.

Important properties of $\mathbb{G}_n$

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Theorem

(Costara, Agler-Young, 2004)

- $\mathbb{G}_2$ *is not biholomorphic to a convex domain,*
- $l_{\mathbb{G}_2} \equiv c_{\mathbb{G}_2}$.

For a function $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ one may define the mapping $F_f(\pi^n(\lambda)) := \pi^n(f(\lambda_1), \dots, f(\lambda_n))$, $\lambda \in \mathbb{D}^n$. Note that $F_f \in \mathcal{O}(\mathbb{G}_n, \mathbb{G}_n)$.

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In particular, we have a description of the group of automorphisms of $\mathbb{G}_n$:

$$\operatorname{Aut}(\mathbb{G}_n) = \{F_a : a \in Aut\mathbb{D}\}. \tag{4}$$

Symmetrized polydisc - more properties, continued

The symmetrized polydisc is an important object when studying the properties of the spectral unit ball

$$\Omega_n := \{A \in \mathbb{C}^{n \times n} : r(A) < 1\}, \tag{5}$$

where $r(A) = \max\{|\zeta| : \zeta \in \operatorname{Spec} A\}$ is the spectral radius of the matrix A.

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$$\det(\lambda I_n - A) = \lambda^n + \sum_{j=1}^{n} (-1)^j \sigma_j(A)\lambda^{n-j}, \tag{6}$$

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gives the natural holomorphic mapping $\sigma : \Omega_n \mapsto \mathbb{G}_n$.The study of $\mathbb{G}_n$ helps understand the form of $\operatorname{Aut} \Omega_n$ (also $\operatorname{Prop}(\Omega_n, \Omega_n)$) and the (regularity of the) spectral Nevanlinna-Pick problem.

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Automorphisms of the spectral ball

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Symmetrized polydisc: higher dimension vs. $n \leq 2$

It was the discovery (in 2005) that the Lempert theorem holds for the symmetrized bidisc that made the domain $\mathbb{G}_2$ (and thus $\mathbb{G}_n$ in general) interesting for geometric function theory.

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Symmetrized polydisc: higher dimension vs. $n \leq 2$

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- $\mathbb{G}_n$ *is a Lu-Qi Keng domain.*

Symmetrized bidisc - very regular domain

It follows from the above description together with the Lempert Theorem that $\mathbb{G}_n$, $n \geq 3$ cannot be exhausted by domains biholomorphic to convex ones. It turned out however that the same holds for $n = 2$.

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The above description leads to the proof of the fact that $\mathbb{G}_2$ may be exhausted by strictly linearly convex domains:
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Tetrablock - properties

Note that similarly as in the case of the polydisc the tetrablock is a proper (of multiplicity two) holomorphic image of a very regular domain (convex and homogeneouous), i. e. the Cartan domain of type two:

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Tetrablock - properties, continued

First a complete decsription of $\operatorname{Aut}\mathbb{E}$ was given (N. J. Young).$\mathbb{E}$ turned out to be non-homogeneuous. The orbit of 0 is the set $\{x \in \mathbb{E} : x_1 x_2 = x_3\}$.
Moreover, all proper holomorphic self-mappings are automorphisms (Ł. Kosiński).Finally, we have.

Theorem

(Edigarian-Kosiński-Z, 2011)

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Tetrablock vs. symmetrized bidisc

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Let $f : (\mathbb{B}_m)^n_{sym} \to (\mathbb{B}_m)^n_{sym}$, $m, n \geq 2$, be a proper holomorphic mapping. Then there is an automorphism g of $\mathbb{B}_m$ such that $f = F_g$.

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Can one say a little more about the geometry (function theory) on $(\mathbb{B}_m)^n_{sym}$?

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We also know that $S_n(D)$ is always linearly convex.

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Another possible generalization would be the study of (holomorphic) functions invariant with respect to some proper holomorphic maps (or some groups of linear isomorphisms) – compare recent results of Aron, Falco, Garcia, Maestre.