In the footsteps of Pythagoras

Quebec City 2018

ODE TO TOM

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whose colleagues thought he was ”da bomb”
Though deep is his head
he confused the letter z with ”zed”
the final proof was done with aplomb

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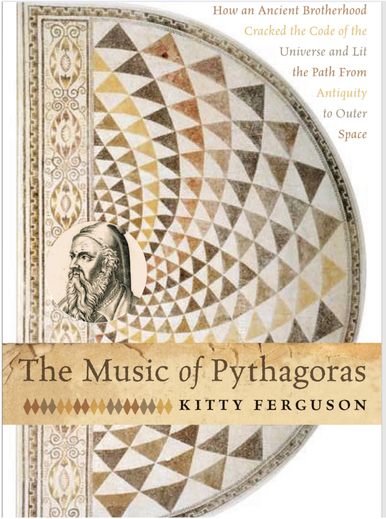
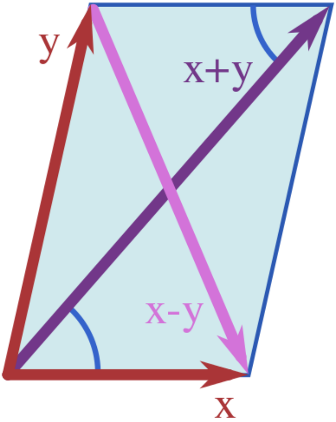
Pythagoras of Samos (570 - 495 BCE)



Book recommendation

Parallelogram law



Found in any linear algebra book

For a Hilbert space $\mathcal{H}$:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\big(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\big), \quad \mathbf{x},\mathbf{y} \in \mathcal{H}.$$

Theorem (P. Jordan/J. von Neumann – 1935)

If $\mathcal{X}$ is a Banach space such that

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then $\mathcal{X}$ is a Hilbert space with

$$\langle \mathbf{x},\mathbf{y}\rangle_{\mathcal{X}} := \tfrac{1}{4}\big(\|\mathbf{x}+\mathbf{y}\|^2 + i\|\mathbf{x}+i\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 - i\|\mathbf{x}-i\mathbf{y}\|^2\big).$$

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WEAK PARALLELOGRAM LAWS

Inspiration

Theorem (Clarkson - 1936)

For $L^p = L^p(\Omega, \Sigma, \mu)$:

$$\|f+g\|_{L^p}^p + \|f-g\|_{L^p}^p \geqslant 2^{p-1}\big(\|f\|_{L^p}^p + \|g\|_{L^p}^p\big), \quad p \in (1,2],$$

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Weak Parallelogram laws

Definition (Bynum, Drew, Cheng, Harris)

A Banach space $\mathcal{X}$ satisfies the *r-lower weak parallelogram law with constant C* ($\mathcal{X}$ is r-LWP(C)), if

$$\|\mathbf{x}+\mathbf{y}\|^r + C\|\mathbf{x}-\mathbf{y}\|^r \leqslant 2^{r-1}\big(\|\mathbf{x}\|^r + \|\mathbf{y}\|^r\big), \quad \mathbf{x},\mathbf{y} \in \mathcal{X}.$$

Similarly, $\mathcal{X}$ satisfies the *r-upper weak parallelogram law with constant C* ($\mathcal{X}$ is r-UWP(C)), if

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Observe that $0 < C \leqslant 1$ in r-LWP(C) and $C \geqslant 1$ in r-UWP(C).

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Theorem (Cheng-Mashreghi-R - 2017)

$$C_{p,r} := \inf_{0 \leqslant t < 1} \frac{2^{r-r/p}(1+t^p)^{r/p} - (1+t)^r}{(1-t)^r}.$$

If $p \in (1,2]$ then L^p is:

r-UWP(1) when $r \in (1,p]$;

r-LWP($C_{p,r}$) when $r \in [2,p']$;

r-LWP(1) when $r \in [p',\infty)$.

If $p \in [2,\infty)$ then L^p is:

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The weak parallelogram constants are optimal.

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Duality

Theorem (Cheng-Harris – 2013)

$$\mathcal{X} \text{ is } p - LWP(C) \iff \mathcal{X}^* \text{ is } p' - UWP(C^{-p'/p}).$$

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ORTHOGONALITY

Hilbert space

Definition

Two vectors $\mathbf{x}, \mathbf{y}$ in a Hilbert space $\mathcal{H}$ are *orthogonal* if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0.$$

$$\ell^2 := \left\{ (a_k)_{k \geqslant 0} : \sum_{k \geqslant 0} |a_k|^2 < \infty \right\}$$

$$\mathbf{a} \perp_{\ell^2} \mathbf{b} \iff \sum_{k \geqslant 0} a_k \overline{b_k} = 0.$$

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Birkhoff-James orthogonality

Definition

Two vectors $\mathbf{x}$ and $\mathbf{y}$ (in this order) in a Banach space $\mathcal{X}$ are *orthogonal in the Birkhoff-James sense*, and write $\mathbf{x} \perp_{\mathcal{X}} \mathbf{y}$, if

$$\|\mathbf{x} + t\mathbf{y}\| \geqslant \|\mathbf{x}\| \quad \forall t \in \mathbb{C}.$$

Theorem (Birkhoff-James - 1947)

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In a Hilbert space we have

$$\mathbf{x} \perp \mathbf{y} \implies \|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

Theorem (Cheng-R. - 2015)

If $\mathcal{X}$ is $p - LWP(C)$, then

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INNER FUNCTIONS

An idea of Beurling

Definition

If $Sf = zf$ on H^2 we say that $J \in H^2 \setminus \{0\}$ is *inner* if

$$J \perp_{H^2} S^n J, \quad n = 1, 2, \cdots.$$

Note that J is inner when

$$0 = \int_0^{2\pi} |J(e^{i\theta})|^2 e^{-in\theta} \frac{d\theta}{2\pi}, \quad n \geqslant 1.$$

So J is inner precisely when $|J|$ is constant (almost everywhere) on $\mathbb{T}$.

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DIY inner function

Take any $f \in H^2 \setminus \{0\}$ and set

$$\widehat{f} = P_{[Sf]} f.$$

Then

$$J := f - \widehat{f} \text{ is inner.}$$

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Inner functions beyond H^2

Definition

$$\ell^p_A := \Big\{ f = \sum_{k \geqslant 0} a_n z^k : (a_k)_{k \geqslant 0} \in \ell^p \Big\}.$$

$$\|f\|_{\ell^p_A} := \|(a_k)_{k \geqslant 0}\|_{\ell^p}.$$

Theorem (Hausdorff-Young)

$$\ell^p_A \subseteq H^{p'}, \quad p \in [1, 2].$$

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Inner functions beyond H^2

Definition

$$\ell^p_A := \Big\{ f = \sum_{k \geqslant 0} a_n z^k : (a_k)_{k \geqslant 0} \in \ell^p \Big\}.$$

$$\|f\|_{\ell^p_A} := \|(a_k)_{k \geqslant 0}\|_{\ell^p}.$$

Theorem (Hausdorff-Young)

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Inner functions in ℓ^p_A

Definition

$J \in \ell^p_A \setminus \{0\}$ is *p-inner* if

$$J \perp_{\ell^p_A} S^n J, \quad n \geqslant 1.$$

DIY p-inner function: $f \in \ell^p_A \setminus \{0\}$ and $\widehat{f}$ is the closest point in $[Sf]$ to f,

$$J = f - \widehat{f} \text{ is } p\text{-inner.}$$

Proposition

$$w \in \mathbb{D} \setminus \{0\}, \;\; f(z) = 1 - \frac{z}{w} \implies J(z) = \frac{1 - z/w}{1 - |w|^{p-2}\overline{w}z}.$$

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Zero sets for ℓ^p_A

Theorem (Cheng-Mashreghi-R - 2018)

Let $p \in (1, \infty)$ and $W = (w_1, w_2, \ldots) \subseteq \mathbb{D} \setminus \{0\}$. Define

$$f_n(z) := \prod_{k=1}^{n} (1 - \frac{z}{w_k}), \quad J_n := f_n - \widehat{f_n}.$$

Then

1. $\|J_n\|_{\ell^p_A}$ *is monotone increasing with* n*;*
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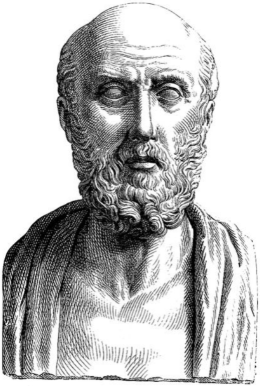

Hippocrates (460 – 370 BCE)

Do no harm

Do no harm to H^2

Suppose $p = 2$, and

$$f_n(z) = \prod_{k=1}^{n} (1 - \frac{z}{w_k}).$$

Then

$$J_n(z) = \Big(\prod_{k=1}^{n} \frac{1}{w_k}\Big) \prod_{k=1}^{n} \frac{w_k - z}{1 - \overline{w_k} z}$$

$(w_k)_{k \geqslant 1}$ is a zero set for H^2

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Speaking of finite Blaschke products.....

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Finite Blaschke Products and Their Connections

Authors: **Garcia**, Stephan Ramon, **Mashreghi**, Javad, **Ross**, William

Explains connections in complex analysis, linear algebra, operator theory, and electrical engineering

SPECIFIC ZERO SETS FOR ℓ^p_A

A useful tool for examples

Proposition

$\|J_n\|_p = \inf\{\|F\|_p : F \in \ell^p_A, F(0) = 1, F(w_k) = 0, 1 \leqslant k \leqslant n\}.$

Taking the theorem for a test drive

Theorem (Cheng-Mashreghi-R (2018))

Let $p \in (1, \infty)$ *and let* $(w_k)_{k \geqslant 1} \subseteq \mathbb{D} \setminus \{0\}$. *Choose* $r_k > 1$ *such that*

$$\sum_{k \geqslant 1} \left(1 - \frac{1}{r_k}\right) < \frac{1}{p'}.$$

If $(w_k)_{k \geqslant 1}$ *satisfies*

$$\sum_{k \geqslant 1} (1 - |w_k|^{r'_k})^{r_k - 1} < \infty,$$

then $(w_k)_{k \geqslant 1}$ *is a zero set for* ℓ^p_A.

Recall that

$$\ell^p_A \subseteq H^{p'}, \quad p \in (1,2]$$

$$\ell^p_A \supseteq H^{p'}, \quad p \in [2,\infty)$$

This means that

- When $p \in (1,2]$, every zero set for ℓ^p_A is a Blaschke sequence
- When $p \in [2,\infty)$, every Blaschke sequence is a zero set for ℓ^p_A

Are the converses true?

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$$F_k(z) := \left(1 - \frac{z}{r_1}\right)\left(1 - \frac{1}{2}\left[\frac{z^{2!}}{r_2^{2!}} + \frac{z^{2\cdot 2!}}{r_2^{2\cdot 2!}}\right]\right)\left(1 - \frac{1}{3}\left[\frac{z^{3!}}{r_3^{3!}} + \frac{z^{2\cdot 3!}}{r_3^{2\cdot 3!}} + \frac{z^{3\cdot 3!}}{r_3^{3\cdot 3!}}\right]\right) \times \cdots \times \left(1 - \frac{1}{k}\left[\frac{z^{k!}}{r_k^{k!}} + \frac{z^{2\cdot k!}}{r_k^{2\cdot k!}} + \frac{z^{3\cdot k!}}{r_k^{3\cdot k!}} + \cdots + \frac{z^{k\cdot k!}}{r_k^{k\cdot k!}}\right]\right)$$

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Theorem (Cheng-Mashreghi-R (2018))

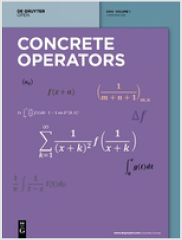
For each $p > 2$, there exists a non-Blaschke sequence $(w_k)_{k\geqslant 1} \subseteq \mathbb{D}$, i.e.,

$$\sum_{k\geqslant 1}(1 - |w_k|) = \infty$$

that is a zero sequence for ℓ^p_A.

DE GRUYTER

CONCRETE OPERATORS



τὴν μεγίστην χάριν οἶδα ὅτι μου ἠκούσατε λέγοντος