Boundary Values of Holomorphic Distributions

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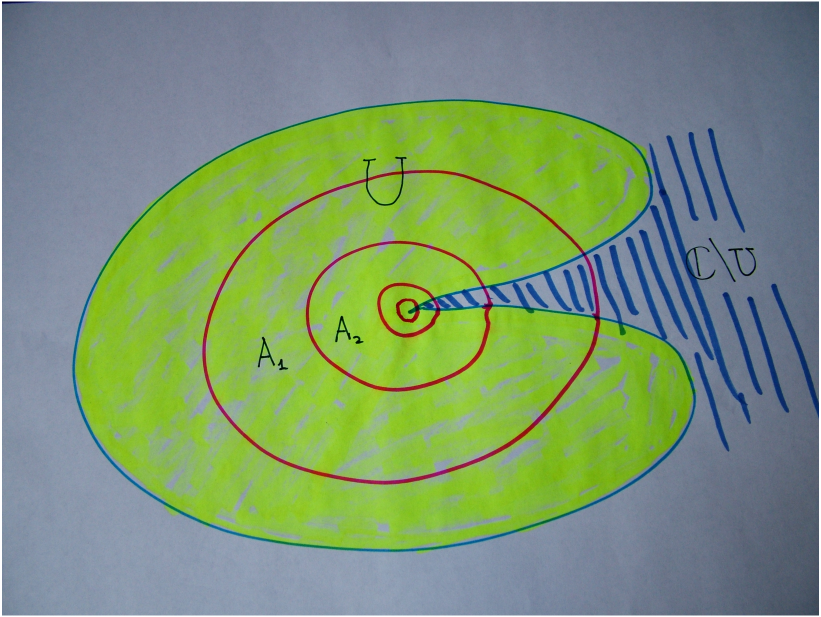
An open set $U \subset \mathbb{C}$ and some boundary points:

Wiener series: $\sum_{n=1}^{\infty} 2^n c(A_n \setminus U)$



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Hedberg (1969): $R^p(X)$: $\leftrightarrow$ a condenser capacity.

Kinds of boundary value

- Concrete: limit taken in some set.
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Abstract version should be sensible on some dense subset of B, and norm-continuous there, so that it has a unique extension to B.
Concept: continuous point evaluation.

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Lipschitz class holomorphic function spaces

Fix $0 < \alpha < 1$, $U \in \mathbb{C}$ open and consider

$$A^{\alpha}(U) := \{f \in \mathrm{lip}(\alpha) : f \text{ is holo on } U\}.$$

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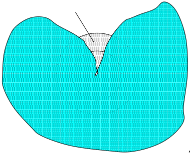
Lord-OF (1991):  $A^\alpha(U)$ admits a continuous point derivation at b if and only if

$$\sum_{n=1}^{\infty} 4^n M_*^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

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Q: Suppose $A^{\alpha}(U)$ admits a continuous point derivation at b, and let ∂ be the normalised derivation there. Is there a set $E \subset U$ such that

$$(*) \lim_{a \to b, a \in E} \frac{f(a) - f(b)}{a - b} = \partial f, \ \forall f \in A?$$

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(1) (2014) If U contains an open triangle with vertex at b, then (*) holds when E is the angle bisector at b.

(2) (2016) In general, (*) holds for some E having full area density at b.

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What is the significance of the condition

$$\sum_{n=1}^{\infty} 2^n M^\beta(A_n \setminus U) < +\infty,$$

when $0 < \beta < 1$, where U is a bounded open subset of $\mathbb{C}$ and $b \in \partial U$?

The answer involves the so-called ‘*negative Lipschitz spaces*’.

Theorem

Let $0 < \beta < 1$ *and* $s = \beta - 1$. *Let* $U \subset \mathbb{C}$ *be a bounded open set, and* $b \in \partial U$. *Then* $A^s(U)$ *admits a continuous point evaluation at* b *if and only if*

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Theorem

Let $0 < \beta < 1$ *and* $s = \beta - 1$. *Let* $U \subset \mathbb{C}$ *be a bounded open set, and* $b \in \partial U$. *Then* $B^s(U)$ *admits a weak-star continuous point evaluation at* b *if and only if*

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It time allows, we shall explain some details of this story, and the conceptual framework around it.

2-D Complex Analysis: on $\mathbb{R}^2$

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Functions analytic on possibly disconnected open sets.

Intrinsic capacities

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More general framework:
(2) (1985) L-F-cap, associated to an elliptic L and a Banach SCS, F.

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- The affine group of $\mathbb{R}^d$ acts by composition on F, and each compact set of affine maps gives an equicontinuous family of composition operators.

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$1f = 0$ on U just means that $U \cap \mathrm{spt}(f) = \emptyset$. The idea is to reduce questions about L and some space F to equivalent problems about 1 and the space LF.

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(4) A theorem that says that in dimension two all SCS are essentially invariant under the Vitushkin localization operators.

The spaces T_s and C_s

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These extend $\mathrm{Lip}(s)_{\mathrm{cs}}$ and $\mathrm{lip}(s)_{\mathrm{cs}}$ to negative S.
Complete them to Banach spaces.

$A^s(U)$ and $B^s(U)$:

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Proof.

Use $\mathfrak{T}_\varphi(f) := \frac{1}{\pi z} * (\varphi \cdot \frac{\partial f}{\partial \bar{z}})$ and a standard pincher. □

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$$N_0(\varphi) \le N_1(\varphi) \le N_2(\varphi) \le \cdots ,$$

$$N_k(\varphi \cdot \psi) \le 2^k N_k(\varphi) N_k(\psi)$$

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A standard pincher $(\varphi_n)_n$ at b has:

- $\varphi_n = 1$ near b,
- $\mathrm{spt}\varphi_n \to \{b\}$,
- $\sup_n N_k(\varphi_n) < +\infty$, for all $k \in \mathbb{N}$.

Estimating $\langle \varphi, f \rangle$

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Version:

$$|\langle L^t \varphi, f \rangle| \overset{?}{\leq} K \cdot \mathbb{N}_k(\varphi) \cdot \|f\|_F \cdot (L\text{-}F\text{-cap})(\mathrm{spt}(\varphi \cdot f)).$$

cf. Vitushkin (d-bar), Mazaloff (Δ).

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Simple example:

$$|\langle \varphi, f \rangle| \leq K \cdot \mathbb{N}_0(\varphi) \cdot \|f\|_{L^\infty} \cdot \text{volume}(\text{spt}(\varphi \cdot f)).$$

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Many SCBS have this strong module property. It implies

$$|\langle \varphi, f \rangle| \leq K \cdot \mathbb{N}_k(\varphi) \cdot \|f\|_F,$$

whenever $\text{spt}(\varphi \cdot f) \in \mathbb{B}(0, 1)$.

Scaling and covering

For $F = T_s$, with $-2 < s < 0$, scaling gives

$$|\langle \varphi, f \rangle| \leq K \cdot \mathbb{N}_k(\varphi) \cdot \|f\|_F \cdot r^{s+2},$$

whenever $\mathrm{spt}(\varphi \cdot f) \in \mathbb{B}(0, r)$.

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and for $f \in C^s$,

$$|\langle \varphi, f \rangle| \leq K \cdot \mathbb{N}_k(\varphi) \cdot \|f\|_F \cdot M_*^{s+2}(\operatorname{spt}(\varphi \cdot f)).$$

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$$|f(0)| \leq K \cdot \sum_{n=1}^{\infty} 2^n \cdot M_*^{\beta}(A_n \setminus U) \cdot \|f\|_s.$$

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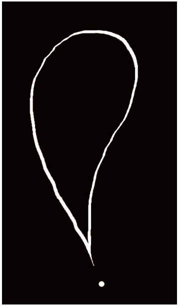
- $h_N(0)$ is unbounded.

Examples



THANKS