An application of entire functions of exponential type in Banach algebras

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"The first joint work with Tom"

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Type

An entire function f is said to be of *exponential type* if it satisfies the growth restriction

$$|f(z)| \leq Ae^{\alpha|z|}, \qquad (z \in \mathbb{C}).$$

The *type* of f is

$$\sigma_f = \inf \alpha = \limsup_{z\to\infty} \frac{\log|f(z)|}{|z|}.$$

Attention: We might have $\sigma_f = 0$, e.g., for polynomials.

Hadamard's Theorem (special case)

Theorem

Let f be an entire function of exponential type such that

$$f(z) \neq 0, \qquad (z \in \mathbb{C}).$$

Then there are constants $\alpha, \beta \in \mathbb{C}$ such that

$$f(z) = e^{\alpha z + \beta}, \qquad (z \in \mathbb{C}).$$

A Uniqueness Result

Let p be a polynomial such that

$$|p(x)| \leq M, \qquad (x \in \mathbb{R}).$$

Then p is constant.

A Uniqueness Result (polynomials growth)

Let p be a polynomial such that

$$|p(x)| \leq M|x|^k, \qquad (x \in \mathbb{R}).$$

Then p is a polynomial of degree at most k.

A Uniqueness Result

Theorem

Let f be an entire function of exponential type zero such that

$$|f(x)| \leq M, \qquad (x \in \mathbb{R}).$$

Then f is constant.

A Uniqueness result

Corollary

Let f be an entire function of exponential type zero such that

$$f(x) = O(|x|^k), \qquad (x \to \pm\infty).$$

Then f is a polynomial of degree at most k.

Maximal Ideal Space

Let $\mathcal{A}$ be a Banach algebra. A nonzero linear functional $\Lambda : \mathcal{A} \longrightarrow \mathbb{C}$ is said to be *multiplicative* if

$$\Lambda(\mathfrak{a}\mathfrak{b}) = \Lambda(\mathfrak{a})\Lambda(\mathfrak{b}), \qquad (\mathfrak{a}, \mathfrak{b} \in \mathcal{A}).$$

It is easy to see that $\ker(\Lambda)$ is a *maximal ideal* of $\mathcal{A}$. Moreover, given any maximal ideal M in $\mathcal{A}$, there is a unique nonzero multiplicative linear functional Λ such that $\ker(\Lambda) = M$.

Theorem (Gleason–Kahane–Zelazko, 1968)

Let $\mathcal{A}$ be a (commutative) Banach algebra. Let $\Lambda : \mathcal{A} \longrightarrow \mathbb{C}$ be a nonzero bounded linear functional on $\mathcal{A}$. Then Λ is multiplicative *if and only if*

$$\Lambda(\mathfrak{a}) \in Sp(\mathfrak{a})$$

for all $\mathfrak{a} \in \mathcal{A}$.

Proof.

Easy direction: If Λ is multiplicative, then $\Lambda(\mathfrak{e}) = 1$ and hence

$$\Lambda\big(\mathfrak{a} - \Lambda(\mathfrak{a})\mathfrak{e}\big) = 0.$$

Therefore, $\mathfrak{a} - \Lambda(\mathfrak{a})\mathfrak{e}$ cannot be invertible, i.e., $\Lambda(\mathfrak{a}) \in \mathrm{Sp}(\mathfrak{a})$.

Proof.

Technical direction: Assume that $\Lambda(\mathfrak{a}) \in \mathrm{Sp}(\mathfrak{a})$ for all $\mathfrak{a} \in \mathcal{A}$. Our goal is to show that Λ is multiplicative.

Fix $\mathfrak{a} \in \mathcal{A}$, and define

$$f(z) = \Lambda(e^{\mathfrak{a}z}), \qquad (z \in \mathbb{C}).$$

Proof.

- f is an entire function of exponential type.
- $f(0) = \Lambda(\mathfrak{e}) \in \mathrm{Sp}(\mathfrak{e}) = \{1\}$. In short, $f(0) = 1$.
- $e^{\mathfrak{a}z}$ is invertible in $\mathcal{A}$. In other words, $0 \notin \mathrm{Sp}(e^{\mathfrak{a}z})$. By the main assumption, $\Lambda(e^{\mathfrak{a}z}) \in \mathrm{Sp}(e^{\mathfrak{a}z})$.

Therefore,

$$f(z) = \Lambda(e^{\mathfrak{a}z}) \neq 0, \qquad (z \in \mathbb{C}).$$

Proof.

Hence, By Hadamard's theorem, there are constants $\alpha, \beta \in \mathbb{C}$ such that

$$f(z) = \Lambda(e^{\mathfrak{a}z}) = e^{\alpha z+\beta}, \qquad (z \in \mathbb{C}).$$

Considering the Taylor coefficients of both sides, and that $f(0) = 1$, we deduce

$$\Lambda(\mathfrak{a}^n) = \alpha^n = (\Lambda(\mathfrak{a}))^n, \qquad (n \geq 1).$$

But, even $\Lambda(\mathfrak{a}^2) = (\Lambda(\mathfrak{a}))^2$ is enough to ensure that Λ is multiplicative. □

Nilpotent Elements

Theorem (Allan, 1996)

Let $\mathfrak{a}$ be an element of a unital Banach algebra, and let $k \geq 0$. Then

$$\|(1+\mathfrak{a})^n - (1-\mathfrak{a})^n\| = O(n^k), \qquad (n \to \infty),$$

if and only if

$$\mathfrak{a}^{k+2} = 0 \quad (k \ odd) \qquad while \qquad \mathfrak{a}^{k+1} = 0 \quad (k \ even).$$

Proof.

‘Only if’: Trivial.

‘If’: Let Λ be a continuous linear functional on the Banach algebra, and define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \Lambda(e^{\mathfrak{a}z} - e^{-\mathfrak{a}z}), \qquad (z \in \mathbb{C}).$$

Hence, f is an entire function of exponential type. The Taylor expansion of f is

$$f(z) = 2 \sum_{\substack{n \,\mathrm{odd} \\ n \geq 0}} \frac{\Lambda(\mathfrak{a}^n)}{n!} z^n.$$

Proof.

Therefore, the type of f is estimated as

$$\sigma_f = \limsup_{\substack{n \text{ odd} \\ n\to\infty}} |\Lambda(\mathfrak{a}^n)|^{1/n} \leq \limsup_{n\to\infty} \|\mathfrak{a}^n\|^{1/n} = r(\mathfrak{a}).$$

An very essential step is to show that

$$\sigma_f = 0.$$

To do so, we show that $r(\mathfrak{a}) = 0$.

Proof.

Let $\lambda \in \mathrm{Sp}(\mathfrak{a})$. Thus,

$$(1+\lambda)^n - (1-\lambda)^n \in \mathrm{Sp}\big((1+\mathfrak{a})^n - (1-\mathfrak{a})^n\big), \qquad (n \geq 1).$$

By assumption,

$$(1+\lambda)^n - (1-\lambda)^n = O(n^k), \qquad (n \to \infty).$$

Therefore, $\lambda = 0$, otherwise the left hand side has exponential growth. In short, $r(\mathfrak{a}) = 0$.

Proof.

For $x \geq 0$,

$$\begin{aligned} e^x|f(x)| &= |\Lambda(e^{x(1+\mathfrak{a})} - e^{x(1-\mathfrak{a})})| \\ &\leq \|\Lambda\| \sum_{n\geq 1} \frac{\|(1+\mathfrak{a})^n - (1-\mathfrak{a})^n\|}{n!} x^n \\ &\leq C \sum_{n\geq 1} \frac{n^k}{n!} x^n \\ &\leq C(1+x^k)e^x. \end{aligned}$$

Proof.

Hence,

$$|f(x)| \leq C(1+|x|^k), \qquad (x \geq 0).$$

As f is an odd function, the same inequality persists for all $x \in \mathbb{R}$.

Proof.

Therefore, by the uniqueness result,

$$f(z) = \Lambda(e^{\mathfrak{a}z} - e^{-\mathfrak{a}z})$$

must be a polynomial of degree at most k. Considering the Taylor coefficients of f, we deduce

$$\Lambda(\mathfrak{a}^{k+2}) = 0 \quad (k \text{ odd}) \qquad \text{while} \qquad \Lambda(\mathfrak{a}^{k+1}) = 0 \quad (k \text{ even}).$$

Since this holds for all functionals Λ, the result follows. □

Subexponential Power Growth

Allan assumed that

$$\|(1+\mathfrak{a})^n - (1-\mathfrak{a})^n\| = O(n^k), \qquad (n \to \infty).$$

What happens if we suppose that, for some $\rho \in (0,1)$,

$$\|(1+\mathfrak{a})^n - (1-\mathfrak{a})^n\| = O(e^{\varepsilon n^\rho}), \qquad (n \to \infty, \forall \varepsilon > 0)?$$

Our Tool

Let $(t_n)_{n\geq 1}$ be a sequence of positive real numbers with $\sum 1/t_n < \infty$. Then we exploit

$$w(z) = \prod_{n\geq 1}\left(1+\frac{z}{t_n}\right). \tag{1}$$

In particular, we need

$$w_\alpha(z) = \prod_{n\geq 1}\left(1+\frac{z}{(n-\frac{1}{2})^{\frac{1}{\alpha}}}\right), \qquad \alpha \in (0,1). \tag{2}$$

Growth of w_α

We know that

$$|w_\alpha(iy)| \sim 2^{-\frac{1}{2\alpha}} \exp\left(\delta\,|y|^\alpha\right), \qquad (y \to \pm\infty),$$

and

$$w_\alpha(x) \sim 2^{-\frac{1}{2\alpha}} \exp\left(\frac{\delta}{\cos(\frac{\pi\alpha}{2})}\, x^\alpha\right), \qquad (x \to +\infty),$$

where

$$\delta = \frac{\pi}{2\sin(\frac{\pi\alpha}{2})}.$$

A Phragmén-Lindelöf principle

Theorem

Let f be an entire function of type zero. Assume that

$$|f(iy)| \leq |w(iy)|, \qquad (y \in \mathbb{R}),$$

where w is defined as in (1). Then

$$|f(z)| \leq w(|z|), \qquad (z \in \mathbb{C}).$$

A Phragmén-Lindelöf principle

Corollary

Let f be an entire function of type zero. Assume that

$$|f(iy)| \leq C \exp(\delta |y|^\alpha), \qquad (y \to \pm\infty).$$

Then

$$|f(z)| \leq C \exp\left\{ \frac{\delta}{\cos(\pi\alpha/2)} |z|^\alpha \right\}, \qquad (z \to \infty).$$

JM–Ransford

Theorem (JM–Ransford, 2004)

Let $\mathfrak{a}$ be an element of a unital Banach algebra, and let $\rho \in (0,1)$. Then the following are equivalent:

(I)

$$\|(1+\mathfrak{a})^n\| \text{ and } \|(1-\mathfrak{a})^n\| = O(e^{\varepsilon n^\rho}), \qquad (n \to \infty, \forall \varepsilon > 0),$$

(II)

$$\|(1+\mathfrak{a})^n - (1-\mathfrak{a})^n\| = O(e^{\varepsilon n^\rho}), \qquad (n \to \infty, \forall \varepsilon > 0),$$

(III)

$$\lim_{n\to\infty} n^{\frac{1}{\rho}-1} \|\mathfrak{a}^n\|^{\frac{1}{n}} = 0.$$

Proof.

$(I) \implies (II)$: Trivial.

$(II) \implies (III)$: Fix $\varepsilon > 0$. Let Λ be a continuous linear functional on the Banach algebra, and define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \Lambda(e^{\mathfrak{a}z} - e^{-\mathfrak{a}z}), \qquad (z \in \mathbb{C}).$$

We know that f is an entire function of exponential type zero.

Proof.

For $x \geq 0$,

$$\begin{aligned} e^x|f(x)| &= |\Lambda(e^{x(1+\mathfrak{a})} - e^{x(1-\mathfrak{a})})| \\ &\leq \|\Lambda\| \sum_{n\geq 0} \frac{\|(1+\mathfrak{a})^n - (1-\mathfrak{a})^n\|}{n!} x^n \\ &\leq C \sum_{n\geq 0} \frac{e^{\varepsilon n^\rho}}{n!} x^n. \end{aligned}$$

Proof.

Since, for $\rho \in (0,1)$ and $\varepsilon > 0$,

$$\sum_{n\geq 0} \frac{e^{\varepsilon n^\rho}}{n!} x^n = O(e^{x+2\varepsilon x^\rho}), \qquad (x \to +\infty),$$

we deduce

$$|f(x)| \leq Ce^{2\varepsilon |x|^\rho}, \qquad (x \geq 0).$$

As f is an odd function, the same inequality persists for all $x \in \mathbb{R}$.

Proof.

Hence, by the corollary, we have

$$|f(z)| \leq C \exp\left(\frac{2\varepsilon}{\cos(\frac{\pi\rho}{2})}|z|^\rho\right), \qquad (z \in \mathbb{C}). \tag{3}$$

Proof.

The Taylor expansion of f is

$$f(z) = 2 \sum_{\substack{n \,\mathrm{odd} \\ n \geq 0}} \frac{\Lambda(\mathfrak{a}^n)}{n!} z^n.$$

The Taylor coefficients can be estimated using the standard Cauchy estimates together with (3).

Proof.

This yields

$$\frac{2|\Lambda(\mathfrak{a}^n)|}{n!} \leq \frac{C}{R^n} \exp\left(\frac{2\varepsilon}{\cos(\frac{\pi\rho}{2})} R^\rho\right), \qquad (n \text{ odd}, \; R > 0).$$

The right-hand side is minimized when

$$\frac{2\varepsilon}{\cos(\frac{\pi\rho}{2})} \rho R^{\rho-1} - nR^{-1} = 0.$$

Proof.

This gives

$$\frac{2|\Lambda(\mathfrak{a}^n)|}{n!} \leq C\left(\frac{2e\varepsilon\rho}{n\cos(\frac{\pi\rho}{2})}\right)^{\frac{n}{\rho}}, \qquad (n \text{ odd}).$$

This is true for each Λ (with a constant C depending on Λ).

Proof.

Magic: By the *uniform boundedness principle*, there exists a universal constant C such that

$$\frac{2\|\mathfrak{a}^n\|}{n!} \leq C\left(\frac{2e\varepsilon\rho}{n\cos(\frac{\pi\rho}{2})}\right)^{\frac{n}{\rho}}, \qquad (n \text{ odd}).$$

Proof.

Take n-th roots, let $n \to \infty$ and use Stirling's formula to obtain

$$\limsup_{\substack{n \text{ odd} \\ n\to\infty}} n^{\frac{1}{\rho}-1} \|\mathfrak{a}^n\|^{1/n} \leq \frac{1}{e}\left(\frac{2e\varepsilon\rho}{\cos(\frac{\pi\rho}{2})}\right)^{\frac{1}{\rho}}.$$

Finally, as ε is arbitrary,

$$\lim_{\substack{n \text{ odd} \\ n\to\infty}} n^{\frac{1}{\rho}-1} \|\mathfrak{a}^n\|^{1/n} = 0,$$

which is part (III).

Proof.

$(III) \implies (I)$: Given $\varepsilon > 0$, choose $\delta > 0$ so that

$$\frac{(e\delta)^\rho}{e\rho} = \frac{\varepsilon}{2}.$$

As $n^{\frac{1}{\rho}-1}\|\mathfrak{a}^n\|^{\frac{1}{n}} < \delta$ for all large enough n, using Stirling's formula again, there exists a constant C such that

$$\frac{\|\mathfrak{a}^n\|}{n!} \le C\frac{(e\delta)^n}{n^{n/\rho}}, \qquad (n \ge 1).$$

Proof.

Therefore,

$$\begin{aligned}
\|(1 \pm \mathfrak{a})^n\| &\leq \sum_{k=0}^{n} \binom{n}{k} \|\mathfrak{a}^k\| \\
&\leq 1 + \sum_{k=1}^{n} n^k \frac{\|\mathfrak{a}^k\|}{k!} \\
&\leq 1 + C \sum_{k=1}^{n} \frac{(ne\delta)^k}{k^{\frac{k}{\rho}}}.
\end{aligned}$$

Proof.

By elementary calculus,

$$\frac{A^k}{k^{\frac{k}{\rho}}} \leq \exp(A^{\frac{\rho}{e\rho}}),$$

for all $A > 0$ and $k \geq 1$. Hence,

$$\|(1 \pm \mathfrak{a})^n\| \leq 1 + Cn \exp \frac{(ne\delta)^\rho}{e\rho} = O(e^{\varepsilon n^\rho}), \qquad (n \to \infty),$$

which is part (I). □

Theorem (Chalendar–Kellay–Ransford, 2000)

Let $\mathfrak{a}$ be an element of a unital Banach algebra, and let $\alpha \in (0, \infty)$ and $\beta \in (1, \infty)$ be such that $\beta^2 - \alpha^2 = 1$. Suppose that

$$\|(1+\mathfrak{a})^n\| \quad \textit{and} \quad \|(1-\mathfrak{a})^n\| = O(\beta^n), \qquad (n \to \infty).$$

Then

$$\|\mathfrak{a}^n\| = O(\alpha^n \log n), \qquad (n \to \infty).$$

Conjecture (Chalendar–Kellay–Ransford, 2000)

Let $\mathfrak{a}$ be an element of a unital Banach algebra, and let $\alpha \in (0, \infty)$ and $\beta \in (1, \infty)$ be such that $\beta^2 - \alpha^2 = 1$. Suppose that

$$\|(1+\mathfrak{a})^n\| \quad \text{and} \quad \|(1-\mathfrak{a})^n\| = O(\beta^n), \qquad (n \to \infty).$$

Then

$$\|\mathfrak{a}^n\| = O(\alpha^n), \qquad (n \to \infty).$$

JM–Ransford

Theorem (JM–Ransford, 2005)

Let $\mathfrak{a}$ be an element of a unital Banach algebra, and let $\alpha \in (0, \infty)$ and $\beta \in (1, \infty)$ be such that $\beta^2 - \alpha^2 = 1$. Then

$$\|(1+\mathfrak{a})^n\| \quad \text{and} \quad \|(1-\mathfrak{a})^n\| = O(\beta^n), \qquad (n \to \infty), \tag{4}$$

if and only if

$$\|\mathfrak{a}^n\| = O(\alpha^n), \qquad (n \to \infty). \tag{5}$$

References

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Thank you