An application of entire functions of exponential type in Banach algebras

Javad Mashreghi Université Laval

"The first joint work with Tom"

Complex Analysis and Spectral Theory Celebrating T. Ransford's 60th Birthday Université Laval May 21-25, 2018

(@) (±) (±

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

Type

An entire function *f* is said to be of *exponential type* if it satisfies the growth restriction

$$
|f(z)| \leq A e^{\alpha |z|}, \qquad (z \in \mathbb{C}).
$$

The *type* of *f* is

$$
\sigma_f = \inf \alpha = \limsup_{z \to \infty} \frac{\log |f(z)|}{|z|}.
$$

Attention: We might have $\sigma_f = 0$, e.g., for polynomials.

メロト メ何ト メミト メミト

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0)

[Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

Hadamard's Theorem (special case)

Theorem

Let f be an entire function of exponential type such that

$$
f(z)\neq 0, \qquad (z\in\mathbb{C}).
$$

Then there are constants $\alpha, \beta \in \mathbb{C}$ *such that*

$$
f(z)=e^{\alpha z+\beta}, \qquad (z\in\mathbb{C}).
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ母メ メミメ メミメ

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

A Uniqueness Result

Let *p* be a polynomial such that

$$
|p(x)| \leq M, \qquad (x \in \mathbb{R}).
$$

Then *p* is constant.

メロメ メ都 メメモ メルモト

重

 $2Q$

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

A Uniqueness Result (polynomials growth)

Let *p* be a polynomial such that

$$
|p(x)| \leq M|x|^k, \qquad (x \in \mathbb{R}).
$$

Then *p* is a polynomial of degree at most *k*.

ALCOHOL:

 $\left\{ \left\vert \left\langle \left\langle \left\langle \mathbf{q} \right\rangle \right\rangle \right\rangle \right\vert \left\langle \mathbf{q} \right\rangle \right\vert \left\langle \mathbf{q} \right\rangle \right\vert \left\langle \mathbf{q} \right\rangle \right\}$

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

A Uniqueness Result

Theorem

Let f be an entire function of exponential type zero such that

 $|f(x)| \leq M$, $(x \in \mathbb{R})$.

Then f is constant.

メロト メ何ト メミト メミト

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

A Uniqueness result

Corollary

Let f be an entire function of exponential type zero such that

$$
f(x) = O(|x|^k), \qquad (x \to \pm \infty).
$$

Then f is a polynomial of degree at most k.

メロメ メ母メ メミメ メミメ

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

Maximal Ideal Space

Let A be a Banach algebra. A nonzero linear functional Λ : A −→ C is said to be *multiplicative* if

$$
\Lambda(\mathfrak{a}\mathfrak{b})=\Lambda(\mathfrak{a})\Lambda(\mathfrak{b}),\qquad (\mathfrak{a},\mathfrak{b}\in\mathcal{A}).
$$

It is easy to see that ker(Λ) is a *maximal ideal* of A. Moreover, given any maximal ideal *M* in A, there is a unique nonzero multiplicative linear functional Λ such that ker(Λ) = M.

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

Theorem (Gleason–Kahane–Zelazko, 1968)

Let A *be a (commutative) Banach algebra. Let* Λ : A −→ C *be a nonzero bounded linear functional on* A*. Then* Λ *is* multiplicative *if and only if*

Λ(a) ∈ *Sp*(a)

for all $a \in A$.

メロト メ何ト メミト メミト

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

Proof.

Easy direction: If Λ is multiplicative, then $\Lambda(\mathfrak{e}) = 1$ and hence

$$
\Lambda\big(\mathfrak{a}-\Lambda(\mathfrak{a})\mathfrak{e}\big)=0.
$$

Therefore, $a - \Lambda(a)\varepsilon$ cannot be invertible, i.e., $\Lambda(a) \in Sp(a)$.

メロメ メ都 メメ きょ メモメ

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0)

[Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

Proof.

Technical direction: Assume that $\Lambda(\mathfrak{a}) \in Sp(\mathfrak{a})$ for all $\mathfrak{a} \in \mathcal{A}$. Our goal is to show that Λ is multiplicative.

Fix $a \in \mathcal{A}$, and define

$$
f(z) = \Lambda(e^{az}), \qquad (z \in \mathbb{C}).
$$

メロト メ何ト メミト メミト

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

Proof.

- *f* is an entire function of exponential type.
- $f(0) = \Lambda(\mathfrak{e}) \in Sp(\mathfrak{e}) = \{1\}.$ In short, $f(0) = 1$.
- e^{az} is invertible in \mathcal{A} . In other words, $0 \not\in Sp(e^{az})$. By the main assumption, $\Lambda(e^{az}) \in Sp(e^{az})$.

Therefore,

$$
f(z) = \Lambda(e^{az}) \neq 0, \qquad (z \in \mathbb{C}).
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ都 メメ きょ メモメ

[Polynomial Power Growth](#page-13-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0) [Elementary Properties](#page-1-0) [Gleason–Kahane–Zelazko Result](#page-7-0)

Proof.

Hence, By Hadamard's theorem, there are constants $\alpha, \beta \in \mathbb{C}$ such that

$$
f(z) = \Lambda(e^{az}) = e^{\alpha z + \beta}, \qquad (z \in \mathbb{C}).
$$

Considering the Taylor coefficients of both sides, and that $f(0) = 1$, we deduce

$$
\Lambda(\mathfrak{a}^n)=\alpha^n=(\Lambda(\mathfrak{a}))^n, \qquad (n\geq 1).
$$

But, even $\Lambda(\mathfrak{a}^2)=(\Lambda(\mathfrak{a}))^2$ is enough to ensure that Λ is multiplicative.

メロメ メ都 メメ きょ メモメ

П

[Nilpotent Elements](#page-13-0)

Nilpotent Elements

Theorem (Allan, 1996)

Let α *be an element of a unital Banach algebra, and let* $k \geq 0$ *. Then*

$$
||(1+\mathfrak{a})^n-(1-\mathfrak{a})^n||=O(n^k), \qquad (n\to\infty),
$$

if and only if

$$
\mathfrak{a}^{k+2} = 0 \quad (k \text{ odd}) \qquad while \qquad \mathfrak{a}^{k+1} = 0 \quad (k \text{ even}).
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ都 メメ きょ メモメ

[Nilpotent Elements](#page-13-0)

Proof.

'Only if': Trivial.

'If': Let Λ be a continuous linear functional on the Banach algebra, and define $f: \mathbb{C} \to \mathbb{C}$ by

$$
f(z) = \Lambda(e^{az} - e^{-az}), \qquad (z \in \mathbb{C}).
$$

Hence, *f* is an entire function of exponential type. The Taylor expansion of *f* is

$$
f(z) = 2 \sum_{\substack{n \text{ odd} \\ n \ge 0}} \frac{\Lambda(\mathfrak{a}^n)}{n!} z^n.
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロト メ何ト メミト メミト

[Nilpotent Elements](#page-13-0)

Proof.

Therefore, the type of *f* is estimated as

$$
\sigma_f = \limsup_{\substack{n \to \infty \\ n \to \infty}} |\Lambda(\mathfrak{a}^n)|^{1/n} \leq \limsup_{n \to \infty} \|\mathfrak{a}^n\|^{1/n} = r(\mathfrak{a}).
$$

An very essential step is to show that

 $\sigma_f = 0.$

To do so, we show that $r(a) = 0$.

[Application of entire functions in Banach algebras](#page-0-0)

メロト メ何ト メミト メミト

[Nilpotent Elements](#page-13-0)

Proof.

Let $\lambda \in Sp(a)$. Thus,

$$
(1+\lambda)^n-(1-\lambda)^n\in\mathrm{Sp}\big((1+\mathfrak{a})^n-(1-\mathfrak{a})^n\big),\qquad (n\geq 1).
$$

By assumption,

$$
(1+\lambda)^n-(1-\lambda)^n=O(n^k), \qquad (n\to\infty).
$$

Therefore, $\lambda = 0$, otherwise the left hand side has exponential growth. In short, $r(a) = 0$.

[Application of entire functions in Banach algebras](#page-0-0)

≮ロト ⊀伊 ▶ ⊀ 君 ▶ ⊀ 君 ▶

Э

[Nilpotent Elements](#page-13-0)

Proof.

For $x \geq 0$, *e* $x + 2x + 3x$

$$
e^{x}|f(x)| = |\Lambda(e^{x(1+a)} - e^{x(1-a)})|
$$

\n
$$
\leq ||\Lambda|| \sum_{n\geq 1} \frac{||(1+a)^{n} - (1-a)^{n}||}{n!}x^{n}
$$

\n
$$
\leq C \sum_{n\geq 1} \frac{n^{k}}{n!}x^{n}
$$

\n
$$
\leq C(1+x^{k})e^{x}.
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ御 メメ きょ メ ヨメー

活

[Nilpotent Elements](#page-13-0)

Proof.

Hence,

$$
|f(x)| \le C(1+|x|^k), \qquad (x \ge 0).
$$

As *f* is an odd function, the same inequality persists for all $x \in \mathbb{R}$.

[Application of entire functions in Banach algebras](#page-0-0)

≮ロト ⊀伊 ▶ ⊀ 君 ▶ ⊀ 君 ▶

重

 $2Q$

[Nilpotent Elements](#page-13-0)

Proof.

Therefore, by the uniqueness result,

$$
f(z) = \Lambda(e^{az} - e^{-az})
$$

must be a polynomial of degree at most *k*. Considering the Taylor coefficients of *f* , we deduce

$$
\Lambda(\mathfrak{a}^{k+2})=0 \quad \text{(k odd)} \qquad \text{while} \qquad \Lambda(\mathfrak{a}^{k+1})=0 \quad \text{(k even)}.
$$

Since this holds for all functionals Λ, the result follows.

メロメ メ母メ メミメ メミメ

П

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Subexponential Power Growth

Allan assumed that

$$
||(1+\mathfrak{a})^n-(1-\mathfrak{a})^n||=O(n^k), \qquad (n\to\infty).
$$

What happens if we suppose that, for some $\rho \in (0,1)$,

$$
||(1+a)^n-(1-a)^n||=O(e^{\varepsilon n^\rho}), \qquad (n\to\infty, \forall \varepsilon>0)?
$$

ALCOHOL:

→ 何 ▶ → 重 ▶ → 重

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Our Tool

Let $(t_n)_{n\geq 1}$ be a sequence of positive real numbers with $\sum 1/t_n < \infty$. Then we exploit

$$
w(z) = \prod_{n\geq 1} \left(1 + \frac{z}{t_n}\right).
$$
 (1)

In particular, we need

$$
w_{\alpha}(z) = \prod_{n\geq 1} \left(1 + \frac{z}{\left(n - \frac{1}{2}\right)^{\frac{1}{\alpha}}}\right), \qquad \alpha \in (0, 1). \tag{2}
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロト メ何ト メミト メミト

 299

Э

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Growth of W_{α}

We know that

$$
|w_{\alpha}(iy)| \sim 2^{-\frac{1}{2\alpha}} \exp(\delta |y|^{\alpha}), \qquad (y \to \pm \infty),
$$

and

$$
w_{\alpha}(x) \sim 2^{-\frac{1}{2\alpha}} \exp\left(\frac{\delta}{\cos(\frac{\pi\alpha}{2})}x^{\alpha}\right), \quad (x \to +\infty),
$$

where

$$
\delta = \frac{\pi}{2\sin(\frac{\pi\alpha}{2})}.
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ都 メメ きょ メモメ

活

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

A Phragmén-Lindelöf principle

Theorem

Let f be an entire function of type zero. Assume that

$$
|f(iy)| \leq |w(iy)|, \qquad (y \in \mathbb{R}),
$$

where w is defined as in [\(1\)](#page-21-1). Then

$$
|f(z)|\leq w(|z|), \qquad (z\in\mathbb{C}).
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロト メ何ト メミト メミト

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

A Phragmén-Lindelöf principle

Corollary

Let f be an entire function of type zero. Assume that

$$
|f(iy)| \leq C \exp(\delta |y|^{\alpha}), \qquad (y \to \pm \infty).
$$

Then

$$
|f(z)| \leq C \exp\left\{\frac{\delta}{\cos(\pi \alpha/2)}|z|^\alpha\right\}, \qquad (z \to \infty).
$$

[Application of entire functions in Banach algebras](#page-0-0)

≮ロト ⊀伊 ▶ ⊀ 君 ▶ ⊀ 君 ▶

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

JM–Ransford

(I)

Theorem (JM–Ransford, 2004)

Let a *be an element of a unital Banach algebra, and let* $\rho \in (0,1)$ *. Then the following are equivalent:*

$$
||(1+a)^n|| \text{ and } ||(1-a)^n|| = O(e^{\varepsilon n^{\rho}}), \qquad (n \to \infty, \forall \varepsilon > 0),
$$

(II)

$$
||(1+a)^n-(1-a)^n||=O(e^{\varepsilon n^{\rho}}), \qquad (n\to\infty, \forall \varepsilon>0),
$$

(III)

$$
\lim_{n\to\infty}n^{\frac{1}{p}-1}\left\|\alpha^n\right\|^{\frac{1}{n}}=0.
$$

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) σ_{α} [Our Characterization](#page-25-0)

Proof.

 $(I) \Longrightarrow (II)$: Trivial.

 $(H) \Longrightarrow (III)$: Fix $\varepsilon > 0$. Let Λ be a continuous linear functional on the Banach algebra, and define $f: \mathbb{C} \to \mathbb{C}$ by

$$
f(z) = \Lambda(e^{az} - e^{-az}), \qquad (z \in \mathbb{C}).
$$

We know that *f* is an entire function of exponential type zero.

メロト メ何ト メミト メミト

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

For $x \geq 0$,

$$
e^{x}|f(x)| = |\Lambda(e^{x(1+a)} - e^{x(1-a)})|
$$

\n
$$
\leq ||\Lambda|| \sum_{n\geq 0} \frac{||(1+a)^{n} - (1-a)^{n}||}{n!}x^{n}
$$

\n
$$
\leq C \sum_{n\geq 0} \frac{e^{\varepsilon n^{\rho}}}{n!}x^{n}.
$$

[Application of entire functions in Banach algebras](#page-0-0)

K ロ > K @ > K 경 > K 경 > X 경

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

Since, for $\rho \in (0,1)$ and $\varepsilon > 0$,

$$
\sum_{n\geq 0}\frac{e^{\varepsilon n^{\rho}}}{n!}x^{n}=O(e^{x+2\varepsilon x^{\rho}}),\qquad (x\to+\infty),
$$

we deduce

$$
|f(x)| \leq Ce^{2\varepsilon|x|^\rho}, \qquad (x \geq 0).
$$

As *f* is an odd function, the same inequality persists for all $x \in \mathbb{R}$.

≮ロト ⊀伊 ▶ ⊀ 君 ▶ ⊀ 君 ▶

重

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

Hence, by the corollary, we have

$$
|f(z)| \leq C \exp\left(\frac{2\varepsilon}{\cos(\frac{\pi\rho}{2})} |z|^\rho\right), \qquad (z \in \mathbb{C}). \tag{3}
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ都 メメ きょ メモメ

G.

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

The Taylor expansion of *f* is

$$
f(z) = 2 \sum_{\substack{n \text{ odd} \\ n \ge 0}} \frac{\Lambda(\mathfrak{a}^n)}{n!} z^n.
$$

The Taylor coefficients can be estimated using the standard Cauchy estimates together with [\(3\)](#page-29-0).

≮ロト ⊀伊 ▶ ⊀ 君 ▶ ⊀ 君 ▶

 $2Q$

[Entire Functions of Exponential Type](#page-1-0) [Subexponential Power Growth](#page-20-0) [Exponential power growth](#page-38-0)

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

This yields

$$
\frac{2|\Lambda(\mathfrak{a}^n)|}{n!} \leq \frac{C}{R^n} \exp\left(\frac{2\varepsilon}{\cos(\frac{\pi \rho}{2})} R^\rho\right), \qquad (n \text{ odd}, R > 0).
$$

The right-hand side is minimized when

$$
\frac{2\varepsilon}{\cos(\frac{\pi\rho}{2})}\,\rho R^{\rho-1}-nR^{-1}=0.
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ御 メメ きょ メ ヨメ

 299

重

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

This gives

$$
\frac{2|\Lambda(\mathfrak{a}^n)|}{n!} \leq C\Big(\frac{2e\varepsilon\rho}{n\cos(\frac{\pi\rho}{2})}\Big)^{\frac{n}{\rho}}, \qquad (n \text{ odd}).
$$

This is true for each Λ (with a constant *C* depending on Λ).

メロメ メ都 メメ きょ メモメ

重

 $2Q$

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

Magic: By the *uniform boundedness principle*, there exists a universal constant *C* such that

$$
\frac{2\|\mathfrak{a}^n\|}{n!}\leq C\Big(\frac{2e\varepsilon\rho}{n\cos(\frac{\pi\rho}{2})}\Big)^{\frac{n}{\rho}},\qquad(n\text{ odd}).
$$

[Application of entire functions in Banach algebras](#page-0-0)

K ロ ▶ K @ ▶ K 경 ▶ K 경 ▶ ...

重

 $2Q$

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

Take *n*-th roots, let $n \to \infty$ and use Stirling's formula to obtain

$$
\limsup_{\substack{n \text{ odd} \\ n \to \infty}} n^{\frac{1}{\rho}-1} \|\mathfrak{a}^n\|^{1/n} \leq \frac{1}{e} \Big(\frac{2e \varepsilon \rho}{\cos(\frac{\pi \rho}{2})} \Big)^{\frac{1}{\rho}}.
$$

Finally, as ε is arbitrary,

$$
\lim_{\substack{n \text{ odd} \\ n \to \infty}} n^{\frac{1}{p}-1} \|\mathfrak{a}^n\|^{1/n} = 0,
$$

which is part (*III*).

[Application of entire functions in Banach algebras](#page-0-0)

≮ロト ⊀伊 ▶ ⊀ 君 ▶ ⊀ 君 ▶

重

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

 $(HI) \Longrightarrow (I)$: Given $\varepsilon > 0$, choose $\delta > 0$ so that

$$
\frac{(e\delta)^{\rho}}{e\rho}=\frac{\varepsilon}{2}.
$$

As $n^{\frac{1}{\rho}-1}\|\mathfrak{a}^n\|^{\frac{1}{n}}<\delta$ for all large enough n , using Stirling's formula again, there exists a constant *C* such that

$$
\frac{\|\mathfrak{a}^n\|}{n!}\leq C\frac{(e\delta)^n}{n^{n/p}},\qquad (n\geq 1).
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ御 メメ きょうくきょう

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

Therefore,

$$
\begin{array}{rcl} \displaystyle \| (1 \pm \mathfrak{a})^n \| & \leq & \displaystyle \sum_{k=0}^n \binom{n}{k} \| \mathfrak{a}^k \| \\ & \leq & 1 + \sum_{k=1}^n n^k \frac{\|\mathfrak{a}^k\|}{k!} \\ & \leq & 1 + C \sum_{k=1}^n \frac{(n \epsilon \delta)^k}{k^{\frac{k}{p}}} . \end{array}
$$

[Application of entire functions in Banach algebras](#page-0-0)

K ロ > K @ > K 경 > K 경 > 시 경

[The Question](#page-20-0) [The Auxiliary Family](#page-21-0) g_{α} [Our Characterization](#page-25-0)

Proof.

By elementary calculus,

$$
\frac{A^k}{k^{\frac{k}{\rho}}} \leq \exp(A^{\frac{\rho}{e\rho}}),
$$

for all $A > 0$ and $k \ge 1$. Hence,

$$
||(1 \pm \mathfrak{a})^n|| \leq 1 + Cn \exp \frac{(n e \delta)^{\rho}}{e\rho} = O(e^{\varepsilon n^{\rho}}), \qquad (n \to \infty),
$$

which is part (*I*).

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ御 メメ きょくきょう

П

重

[The Conjecture](#page-38-0) [An Affirmative Answer](#page-40-0)

Theorem (Chalendar–Kellay–Ransford, 2000)

Let a *be an element of a unital Banach algebra, and let* $\alpha \in (0,\infty)$ and $\beta \in (1,\infty)$ be such that $\beta^2 - \alpha^2 = 1$. Suppose that

$$
||(1+\mathfrak{a})^n|| \quad \text{and} \quad ||(1-\mathfrak{a})^n|| = O(\beta^n), \qquad (n \to \infty).
$$

Then

$$
\|\mathfrak{a}^n\| = O(\alpha^n \log n), \qquad (n \to \infty).
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロト メ何ト メミト メミト

[The Conjecture](#page-38-0) [An Affirmative Answer](#page-40-0)

Conjecture (Chalendar–Kellay–Ransford, 2000)

Let a *be an element of a unital Banach algebra, and let* $\alpha \in (0,\infty)$ and $\beta \in (1,\infty)$ be such that $\beta^2 - \alpha^2 = 1$. Suppose that

$$
||(1+\mathfrak{a})^n|| \quad \text{and} \quad ||(1-\mathfrak{a})^n|| = O(\beta^n), \qquad (n \to \infty).
$$

Then

$$
\|\mathfrak{a}^n\|=O(\alpha^n),\qquad(n\to\infty).
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロト メ何ト メミト メミト

[The Conjecture](#page-38-0) [An Affirmative Answer](#page-40-0)

JM–Ransford

Theorem (JM–Ransford, 2005)

Let a *be an element of a unital Banach algebra, and let* $\alpha \in (0, \infty)$ and $\beta\in(1,\infty)$ be such that $\beta^2-\alpha^2=1.$ Then

$$
||(1+a)^n|| \quad \text{and} \quad ||(1-a)^n|| = O(\beta^n), \qquad (n \to \infty), \qquad (4)
$$

if and only if

$$
\|\mathfrak{a}^n\| = O(\alpha^n), \qquad (n \to \infty). \tag{5}
$$

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ都 メメ きょ メモメ

[The Conjecture](#page-38-0) [An Affirmative Answer](#page-40-0)

References

- G. Allan, *Sums of idempotents and a lemma of N. J. Kalton*, Studia Math. 121 (1996), 185–191.
- I. Chalendar, K. Kellay, T. Ransford, *Binomial sums, moments and invariant subspaces*, Israel J. Math. 115 (2000), 303–320.
- J. Mashreghi and T. Ransford, *Using entire functions to analyse power growth*, Contemp. Math. 263 (2004), 235–240.
- J. Mashreghi and T. Ransford, *Binomial sums and functions of exponential type*, Bull. London Math. Soc., 37 (2005), 15–24.

メロメ メ都 メメ きょ メモメ

[The Conjecture](#page-38-0) [An Affirmative Answer](#page-40-0)

CMS Summer Meeting, Fredericton, 2003

[Application of entire functions in Banach algebras](#page-0-0)

◀ ㅁ ▶ ◀ @ ▶ ◀ 至 ▶ ◀ 돋

 299

D

[The Conjecture](#page-38-0) [An Affirmative Answer](#page-40-0)

[Application of entire functions in Banach algebras](#page-0-0)

メロメ メ御き メモ ビメモド

重