Multivariate polynomial approximation and convex bodies

Outline

1. *Part I*: Approximation by *holomorphic* polynomials in $\mathbb{C}^d$; Oka-Weil and Bernstein-Walsh
2. *Part 2*: What is *degree* of a polynomial in $\mathbb{C}^d$, $d > 1$? Bernstein-Walsh, revisited
3. *Part 3*: $P-$extremal functions in $\mathbb{C}^d$ (P a convex body in $(\mathbb{R}^+)^d$)

Mostly joint work with L. Bos; based on joint work with T. Bayraktar and T. Bloom.

Part I: Approximation by *holomorphic* polynomials

Let $\mathcal{P}_n$ denote the space of *holomorphic* polynomials of degree at most n in $\mathbb{C}^d$, $d \geq 1$. Let f be a continuous *complex-valued function* on a compact set $K \subset \mathbb{C}^d$. Let

$$\widehat{K} := \{z \in \mathbb{C}^d : |p(z)| \leq ||p||_K = \max_{\zeta \in K} |p(\zeta)|, \text{ all } p \in \cup_n \mathcal{P}_n\}.$$

Theorem

(Oka-Weil) *If $K = \widehat{K}$, then any f holomorphic on a neighborhood of K can be approximated uniformly on K by holomorphic polynomials.*

For $d = 1$, $K = \widehat{K}$ if and only if K has connected complement (non-example: $K = T := \{z : |z| = 1\}$) and this is a version of Runge's theorem.

Sketch of Oka-Weil: Oka's proof

Let f be holomorphic on a neighborhood U of $K \subset \mathbb{C}^d$ (i.e., $f : U \to \mathbb{C}$ is holomorphic in each variable separately).

1. If $K = \widehat{K} \subset \Delta^d := \{z : |z_j| < 1,\ j = 1, ..., d\}$, can approximate K by *polynomial polyhedron*:

$$\Pi = \{z : |z_j| \leq 1,\ j = 1, ..., d,\ |p_k(z)| \leq 1,\ k = 1, ..., l\},$$

   p_k polynomials; i.e., $K \subset \Pi \subset U$.

2. "Lift" problem to $\Delta^{d+l} \subset \mathbb{C}^{d+l}$: we consider $z \to (z, p_1(z), ..., p_l(z))$, i.e., the graph of $(p_1, ..., p_l)$. Using elementary $\bar{\partial}-$theory on polynomial polyhedra, get F holomorphic in neighborhood of Δ^{d+l} with $F(z, p_1(z), ..., p_l(z)) = f(z),\ z \in \Pi$ (Oka extension theorem).

3. Expand F in Taylor series and truncate.

Note: Runge and Oka-Weil are not *quantitative*.

Quantitative Runge and Oka-Weil: Bernstein-Walsh

A result quantifying the Runge and Oka-Weil theorems is the *Bernstein-Walsh theorem*. The key tool is an *extremal function*:

$$V_K(z) := \sup\{u(z) : u \in L(\mathbb{C}^d),\ u \leq 0 \text{ on } K\}$$

$$= \max[0, \sup\{\frac{1}{deg(p)} \log|p(z)| : ||p||_K := \max_{\zeta\in K}|p(\zeta)| \leq 1\}]$$

where $L(\mathbb{C}^d) := \{u \in PSH(\mathbb{C}^d) : u(z) - \log|z| = 0(1),\ |z| \to \infty\}$ and $p \in \cup_n \mathcal{P}_n$. Here $K \subset \mathbb{C}^d$ is compact. For $p_n \in \mathcal{P}_n$,

$$|p_n(z)| \leq ||p_n||_K \exp(nV_K(z)),\ z \in \mathbb{C}^d. \qquad (BW)$$

1. $K = \{z : |z-a| \leq r\}$: $V_K(z) = \log^+ \frac{|z-a|}{r} := \max[0, \log \frac{|z-a|}{r}]$

   et $|p_n(z)| \leq ||p_n||_K \cdot |z-a|^n$ pour $|z-a| > r$.

2. $K = [-1,1] \subset \mathbb{C}$: $V_K(z) = \log|z + \sqrt{z^2-1}|$.

Bernstein-Walsh

For a continuous complex-valued function f on K, let

$$D_n(f, K) := \inf\{||f - p_n||_K : p_n \in \mathcal{P}_n\}.$$

Theorem

Let $K \subset \mathbb{C}^d$ $(d \geq 1)$ be compact with V_K continuous. Let $R > 1$, and let $\Omega_R := \{z : V_K(z) < \log R\}$. Let f be continuous on K. Then

$$\limsup_{n\to\infty} D_n(f, K)^{1/n} \leq 1/R$$

if and only if f is the restriction to K of F holomorphic in Ω_R.

Easy direction: $\limsup_{n\to\infty} D_n^{1/n} \leq 1/R$ for some $R > 1$

To show f extends hol. to Ω_R, take p_n with $D_n = ||f - p_n||_K$.
Claim: $p_0 + \sum_1^\infty (p_n - p_{n-1})$ converges locally uniformly on Ω_R to a holomorphic function F which agrees with f on K.
Proof: Fix R' with $1 < R' < R$; by hypothesis $||f - p_n||_K \leq \frac{M}{R'^n}$ for some $M > 0$. Fix $1 < \rho < R'$, and apply (BW) to the polynomial $p_n - p_{n-1} \in \mathcal{P}_n$ on $\bar{\Omega}_\rho$ to obtain

$$\sup_{\bar{\Omega}_\rho} |p_n(z) - p_{n-1}(z)| \leq \rho^n ||p_n - p_{n-1}||_K$$

$$\leq \rho^n (||p_n - f||_K + ||f - p_{n-1}||_K) \leq \rho^n \frac{M(1 + R')}{R'^n}.$$

Since ρ and R' were arbitrary with $1 < \rho < R' < R$, $p_0 + \sum_1^\infty (p_n - p_{n-1})$ converges locally uniformly on Ω_R to a holomorphic function F. *Converse requires (pluri-)potential theory*.

Converse proof of BW in $\mathbb{C}$ using polynomial interpolation

Let $\Gamma \subset \mathbb{C}$ be a rectifiable Jordan curve with $z_1, \ldots, z_n$ inside Γ, and let f be holomorphic inside and on Γ. Let $L_{n-1}(z)$ be the *Lagrange interpolating polynomial* (LIP) associated to $f, z_1, \ldots, z_n$: $L_{n-1} \in \mathcal{P}_{n-1}$ and $L_{n-1}(z_j) = f(z_j),\ j = 1, \ldots, n$.

Proposition

(Hermite Remainder Formula) For any z inside Γ,

$$f(z) - L_{n-1}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega_n(z)}{\omega_n(t)} \frac{f(t)}{(t-z)}\, dt,$$

where $\omega_n(z) = \prod_{k=1}^n (z - z_k)$.

Take $z_1^{(n)}, \ldots, z_n^{(n)}$ Fekete points of order $n-1$ for K: ω_n satisfies

$$\frac{1}{n} \log\left(\frac{|\omega_n(z)|}{||\omega_n||_K}\right) \to V_K(z) \text{ loc. unif. in } \mathbb{C} \setminus \widehat{K}.$$

For f holomorphic in Ω_R take LIP's for $f, z_1^{(n)}, \ldots, z_n^{(n)}$; $\Gamma \approx \partial\Omega_R$.

Step 1: Let $b_n = \dim \mathcal{P}_n$ and $Q_1, ..., Q_{b_n}$ be a basis for $\mathcal{P}_n$. Define

$$D_R := \{z \in \mathbb{C}^d : |Q_j(z)| < R^n, \ j = 1, ..., b_n\}.$$

Proposition

Let f be holomorphic in a neighborhood of $\bar{D}_R$. Then for each positive integer m, there exists $G_m \in \mathcal{P}_m$ such that for all $\rho \leq R$,

$$||f - G_m||_{\bar{D}_\rho} \leq B(\rho/R)^m$$

where B is a constant independent of m.

Proof: Use a version of "lifting" result of Oka. Let $S : \mathbb{C}^d \to \mathbb{C}^{b_n}$ via $S(z) := (Q_1(z), ..., Q_{b_n}(z))$. Then $S(\mathbb{C}^d)$ is a subvariety of $\mathbb{C}^{b_n}$ and $S(D_R)$ is a subvariety of the polydisk

$$\Delta_R := \{\zeta \in \mathbb{C}^{b_n} : |\zeta_j| < R^n, \ j = 1, ..., b_n\}.$$

Oka: Get F holomorphic in $\Delta_R \subset \mathbb{C}^{b_n}$ with $F \circ S = f$ on D_R.

Step 2: Construct polynomials $Q_1, ..., Q_{b_n}$ so that for n large the sets D_R approximate the sublevel sets Ω_R of V_K (for $0 < R_1 < R$, there exists n_0 such that for all $n \geq n_0$, $D_{R_1} \subset \bar{\Omega}_R \subset \bar{D}_R$). We use *Fekete points* $\{z_j^{(n)}\}$ *of order* n *for* K to construct *fundamental Lagrange interpolating polynomials* $Q_j := l_j^{(n)} \in \mathcal{P}_n$; so $l_j^{(n)}(z_k^{(n)}) = \delta_{jk}$ and $||l_j^{(n)}||_K = 1$. More later ...
Step 3: For any $R' < R$ the Lagrange interpolating polynomials $L_n(f)$ for f associated with a Fekete array for K satisfy

$$||f - L_n(f)||_K \leq B/(R')^n$$

where B is a constant independent of n.

Part 2: What is *degree* of a polynomial in $\mathbb{C}^d$, $d > 1$?

Above, the degree of a polynomial is its *total* degree; i.e., if

$$p(z) = \sum c_\alpha z^\alpha, \; c_\alpha \in \mathbb{C}, \; z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d},$$

the total degree of p is its maximal l^1-degree:

$$\max\{|\alpha| := \alpha_1 + \cdots + \alpha_d : c_\alpha \neq 0\}.$$

Trefethen (PAMS, 2017) considered $D_n(f, K)$ for l^1-degree (*total* degree), l^2-degree (*Euclidean* degree) or l^∞-degree (*max* degree). He compared these three (numerically) for $K = [-1, 1]^d$ and f a multivariate Runge-type function, e.g.,

$$f(z) := \frac{1}{r^2 + \sum_{j=1}^d z_j^2}, \quad r > 0.$$

This f is holomorphic in a neighborhood of K in $\mathbb{C}^d$. We explain *exactly* his numerical conclusions.

The right framework

Fix a *convex body* $P \subset (\mathbb{R}^+)^d$*; i.e., P is compact, convex and* $P^o \neq \emptyset$ (e.g., P is a non-degenerate convex polytope, i.e., the convex hull of a finite subset of $(\mathbb{Z}^+)^d$ in $(\mathbb{R}^+)^d$ with nonempty interior). We consider the finite-dimensional polynomial spaces

$$Poly(nP) := \{p(z) = \sum_{J \in nP \cap (\mathbb{Z}^+)^d} c_J z^J : c_J \in \mathbb{C}\}$$

for $n = 1, 2, \ldots$ where $z^J = z_1^{j_1} \cdots z_d^{j_d}$ for $J = (j_1, \ldots, j_d)$. We let $\mathbf{b_n}$ be the dimension of $Poly(nP)$. For $P = \Sigma$ where

$$\Sigma := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \sum_{j=1}^{d} x_i \leq 1\},$$

we have $Poly(n\Sigma) = \mathcal{P}_n$, the usual space of holomorphic polynomials of degree at most n in $\mathbb{C}^d$. We assume that $\Sigma \subset kP$ for some $k \in \mathbb{Z}^+$: note $0 \in P$ so $Poly(nP)$ are linear spaces.

Convexity and remark on degree

Convexity of P implies that

$$p_n \in Poly(nP),\ p_m \in Poly(mP) \Rightarrow p_n \cdot p_m \in Poly((n+m)P).$$

We may define an associated "norm" for $x = (x_1, ..., x_d) \in \mathbb{R}_+^d$ via

$$\|x\|_P := \inf_{\lambda>0} \{x \in \lambda P\}$$

and define a general degree of a polynomial $p(z) = \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha z^\alpha$ associated to the convex body P as

$$\deg_P(p) := \max_{a_\alpha \neq 0} \|\alpha\|_P.$$

Then $Poly(nP) = \{p : \deg_P(p) \leq n\}$.

Trefethen degree: PAMS, 2017

For $q \geq 1$, if we let

$$P_q := \{(x_1, ..., x_d) : x_1, ..., x_d \geq 0,\ x_1^q + \cdots + x_d^q \leq 1\}$$

be the $(\mathbb{R}^+)^d$ portion of an ℓ^q ball then we have, in Trefethen's notation (on the left),

$$\begin{aligned} d_T(p) &:= \deg_{P_1}(p) \quad (\textit{total} \text{ degree}); \\ d_E(p) &:= \deg_{P_2}(p) \quad (\textit{Euclidean} \text{ degree}); \\ d_{\max}(p) &:= \deg_{P_\infty}(p) \quad (\textit{max} \text{ degree}). \end{aligned}$$

Example

$p(z_1, z_2) = z_1^2 z_2^3$ has $\deg_{P_1}(p) = 5$; $\deg_{P_2}(p) = \sqrt{13}$; $\deg_{P_\infty}(p) = 3$ so $p \in Poly(5P_1) \cap Poly(4P_2) \cap Poly(3P_\infty)$.

We fix a convex body $P \subset (\mathbb{R}^+)^d$ with $\Sigma \subset kP$ for some $k \in \mathbb{Z}^+$.

Bernstein-Walsh, revisited

For $K \subset \mathbb{C}^d$ compact and "nonpluripolar" and f continuous on K,

$$V_{P,K}(z) := \lim_{n\to\infty} [\sup\{\frac{1}{n}\log|p_n(z)| : \ p_n \in Poly(nP), \ ||p_n||_K \leq 1\}]$$

$$\text{and } D_n(f,P,K) := \inf\{||f - p_n||_K : p_n \in Poly(nP)\}.$$

Theorem

(Bos-L.) *Let K be compact with $V_{P,K}$ continuous. Let $R > 1$ and*

$$\Omega_R := \Omega_R(P,K) = \{z : V_{P,K}(z) < \log R\}.$$

Let f be continuous on K. Then

$$\limsup_{n\to\infty} D_n(f,P,K)^{1/n} \leq 1/R.$$

if and only if f is the restriction to K of F holomorphic in Ω_R.

The proof that $\limsup_{n\to\infty} D_n(f, P, K)^{1/n} \leq 1/R$ implies f is the restriction to K of a function holomorphic in Ω_R works exactly as before. For the converse direction, the proof is a modification of the case $P = \Sigma$:

1. use a version of the "lifting" result of Oka on D_R-sets
2. construct basis $\{Q_j\}$ for $Poly(nP)$ so D_R approximate Ω_R
3. Use Lagrange interpolating polynomials $L_n(f)$ for f at $P-$*Fekete points* of K.

Remark: The "true" definition of $V_{P,K}$ will be given later.

We will give some examples first and discuss $P-$Fekete points (and more general $P-$pluripotential theory) later.

Examples: $V_{P,K}$ for product sets

For $P \subset \mathbb{R}^d$ a convex body, the *indicator function* is

$$\phi_P(x_1, ..., x_d) := \sup_{(y_1,...,y_d)\in P} (x_1 y_1 + \cdots x_d y_d).$$

Proposition

Let $E_1, ..., E_d \subset \mathbb{C}$ be compact with V_{E_j} continuous. Then

$$V_{P,E_1\times\cdots\times E_d}(z_1, ..., z_d) = \phi_P(V_{E_1}(z_1), ..., V_{E_d}(z_d)).$$

We use a *global domination principle* (TBD later). In particular, for $K = T^d = \{(z_1, ..., z_d) : |z_1| = \cdots = |z_d| = 1\}$, the unit $d-$torus in $\mathbb{C}^d$,

$$V_{P,T^d}(z) = H_P(z) := \max_{J\in P} \log |z^J| = \phi_P(\log^+ |z_1|, ..., \log^+ |z_d|)$$

(*logarithmic* indicator function of P). Here $|z^J| := |z_1|^{j_1} \cdots |z_d|^{j_d}$.

Trefethen Runge-type example

Let $1/q' + 1/q = 1$ so $\phi_{P_q}(x) = ||x||_{\ell_{q'}}$. If $E_1, ..., E_d \subset \mathbb{C}$,

$$V_{P_q, E_1 \times \cdots \times E_d}(z_1, ..., z_d) = \|[V_{E_1}(z_1), V_{E_2}(z_2), \cdots V_{E_d}(z_d)]\|_{\ell_{q'}}$$
$$= [V_{E_1}(z_1)^{q'} + \cdots + V_{E_d}(z_d)^{q'}]^{1/q'}.$$

For the particular product set, $K := [-1,1]^d$ where $E_j = [-1,1]$ for $j = 1, ..., d$, that Trefethen considers, $V_{E_j}(z_j) = \log|z_j + \sqrt{z_j^2 - 1}|$ and hence we have for $q > 1$

$$V_{P_q, [-1,1]^d}(z_1, ..., z_d) = \left\{ \sum_{j=1}^{d} \left(\log \left| z_j + \sqrt{z_j^2 - 1} \right| \right)^{q'} \right\}^{1/q'}.$$

For $f(z) := \frac{1}{r^2 + z_1^2 + \cdots + z_d^2}$ and $K = [-1,1]^d$, f is holomorphic except on its singular set $S = \{z \in \mathbb{C}^d : \sum_{j=1}^{d} z_j^2 = -r^2\}$, an algebraic variety having no real points.

Trefethen Runge-type example, cont'd

Note $V_{P_{1,[-1,1]^d}}(z_1, ..., z_d) = \max_{j=1,...,d} \log |z_j + \sqrt{z_j^2 - 1}|$. By the theorem, the $D_n(f, P, K)$ decay like $\frac{1}{R^n}$ where

$$R = R(P, K) := \sup\{R' > 0 \,:\, \Omega_{R'} \cap S = \emptyset\}.$$

Clearly $\log(R(P, K)) = \min_{z \in S} V_{P,K}(z)$. We show (Bos.-L):

$$R(P_1, K) = r/\sqrt{d} + \sqrt{1 + r^2/d}$$

$$< r + \sqrt{1 + r^2} = R(P_2, K) = R(P_\infty, K) \text{ (indep. of } d\text{!)}.$$

Thus the approximation order of the Euclidean degree is considerably higher than for the total degree, while the use of max degree provides no additional advantage, as reported by Trefethen.

Unfair!

This is not fair: the dimensions of $\{p : \deg_P(p) \le n\}$ are proportional (asymptotically) to the volume $\mathrm{vol}_d(P)$:

$$\dim(\{p : \deg_{\mathrm{P}}(p) \le n\}) = \dim(Poly(nP)) \asymp \mathrm{vol}_d(P) \cdot n^d.$$

To equalize their dimensions we scale P_q by

$$c = c(q) = \left(\frac{\mathrm{vol}_d(P_1)}{\mathrm{vol}_d(P_q)}\right)^{1/d}$$

and observe that $R(cP, K) = (R(P, K))^c$. For $d = 2$ we compare

$$R(P_1, K) = r/\sqrt{2} + \sqrt{1 + r^2/2},\ R(P_2, K)^{c(2)} = \left(r + \sqrt{1 + r^2}\right)^{\sqrt{2/\pi}}.$$

We have $R(P_2, K)^{c(2)} > R(P_1, K)$ for "small" r ($r < 2.1090\cdots$); so Euclidean degree, even normalized, has a better approximation order than the total degree case, but now with a lesser advantage.

Conclusion and questions

The bottom line is: the Bernstein-Walsh type theorem shows that the geometry of the singularities of the approximated function f – provided f is, indeed, holomorphic on a neighborhood of K! – relative to the sublevel sets of the $P-$extremal function $V_{P,K}$ govern the asymptotics of the sequence $\{D_n(f,P,K)\}$ and hence the number $R(P,K)$ (which we should write as $R(P,K,f)$).

Given K,P,f, let $\Omega_{R(P,K)} := \{z : V_{P,K}(z) < \log R(P,K)\}$ where $\limsup_{n\to\infty} D_n(f,P,K)^{1/n} = 1/R(P,K)$. *Questions*:

1. Given K,f find the "best" P (with equalized $\dim(Poly(nP))$) so that $R(P,K)$ is largest.
2. If $\limsup_{n\to\infty} D_n(f,K)^{1/n} = \limsup_{n\to\infty} D_n(g,K)^{1/n} = 1/R$, then f,g are holomorphic in $\Omega_R = \{z : V_K(z) < \log R\}$ (classical Bernstein-Walsh). Which is "better"?

Example

Example

In $\mathbb{C}$, let $K = \{z : |z| \leq 1/R\}$. Then $f(z) = \frac{1}{z-1}$ is "better" than $g(z) = \sum z^{n!}$ as holomorphic functions in the unit disk Ω_R.

We can't "detect" this, but in $\mathbb{C}^d$, $d > 1$, given K, f, we can compute $\Omega_{R(P,K)}$ for *various* convex bodies P to get a better picture of the true "region of holomorphicity" of f: it contains $\cup_P \Omega_{R(P,K)}$. For "real" theory, we can look, e.g., at traces $\Omega_{R(P,K)} \cap \mathbb{R}^d$ for f real-analytic on a neighborhood of $K \subset \mathbb{R}^d$.

We return to the proof of the "hard" direction of the theorem: to show that f holomorphic in $\Omega_R = \{z : V_{P,K}(z) < \log R\}$ implies

$$\limsup_{n\to\infty} D_n(f, P, K)^{1/n} \leq 1/R.$$

Sketch of proof

Step 1: Let $\mathbf{b_n}$ =dim$Poly(nP)$. Let $Q_1, ..., Q_{\mathbf{b_n}}$ be a basis for $Poly(nP)$. Define

$$D_R := \{z \in \mathbb{C}^d : |Q_j(z)| < R^n, \ j = 1, ..., \mathbf{b_n}\}.$$

Proposition

Let f be holomorphic in a neighborhood of $\bar{D}_R$. Then for each positive integer m, there exists $G_m \in Poly(mP)$ such that for all $\rho \leq R$,

$$||f - G_m||_{\bar{D}_\rho} \leq B(\rho/R)^m$$

where B is a constant independent of m.

Proof follows $P = \Sigma$ case and requires $S : \mathbb{C}^d \to \mathbb{C}^{\mathbf{b_n}}$ via $S(z) := (Q_1(z), ..., Q_{\mathbf{b_n}}(z))$ be one-to-one on $\bar{D}_R$ (follows for $n \geq k$ where $\Sigma \subset kP$).

Sketch of proof, continued

Step 2: Construct polynomials $Q_1, ..., Q_{\mathbf{b_n}}$ so that for n large the sets D_R approximate the sublevel sets Ω_R of $V_{P,K}$; i.e., given $0 < R_1 < R$, there exists n_0 such that for all $n \geq n_0$, $D_{R_1} \subset \bar{\Omega}_R$. Use $P-$*Fekete points* and *fundamental Lagrange interpolating polynomials* to construct Q_j. Here $V_{P,K}(z) = \lim_{n\to\infty} \frac{1}{n} \log \Phi_{P,n}(z)$ where $\Phi_{P,n}(z) := \sup\{|p_n(z)| : \ p_n \in Poly(nP), \ ||p_n||_K \leq 1\}$ and if $V_{P,K}(z)$ is continuous, the convergence is locally uniform on $\mathbb{C}^d$.

Step 3: For any $R' < R$ the Lagrange interpolating polynomials $L_n(f)$ for f associated with a $P-$Fekete array for K satisfy

$$||f - L_n(f)||_K \leq B/(R')^n$$

where B is a constant independent of n.

Part 3: $P-$extremal functions in $\mathbb{C}^d$

We briefly describe some basics of the "$P-$ pluripotential theory." Associated to $P \subset (\mathbb{R}^+)^d$ a convex body, recall we have the *logarithmic indicator function* on $\mathbb{C}^d$

$$H_P(z) := \sup_{J \in P} \log |z^J| := \sup_{J \in P} \log[|z_1|^{j_1} \cdots |z_d|^{j_d}];$$

$$L_P = L_P(\mathbb{C}^d) := \{u \in PSH(\mathbb{C}^d) : u(z) - H_P(z) = 0(1), \ |z| \to \infty\}.$$

For $p \in Poly(nP)$, $n \geq 1$ we have $\frac{1}{n} \log |p| \in L_P$. Given $K \subset \mathbb{C}^d$, the *$P-$extremal function of K* is $V_{P,K}(z)$ where **the a priori definition of $V_{P,K}$ is**

$$V_{P,K}(z) := \sup\{u(z) : u \in L_P(\mathbb{C}^d), \ u \leq 0 \text{ on } K\}.$$

This is a *Perron-Bremmerman* family: $(dd^c V_{P,K})^d = 0$ outside K.
Example: $K = T^d$, the unit $d-$torus in $\mathbb{C}^d$. Then

$$V_{P,T^d}(z) = H_P(z) = \sup_{J \in P} \log |z^J|.$$

Example: $V_{P,T^d}(z) = H_P(z) = \sup_{J\in P} \log |z^J|$

Here, $(dd^c V_{P,T^d})^d = \frac{d!Vol(P)}{(2\pi)^d} d\theta_1 \cdots d\theta_d$ is the *Monge-Ampère measure* of V_{P,T^d}. For a C^2-function u on $\mathbb{C}^d$,

$$(dd^c u)^d = dd^c u \wedge \cdots \wedge dd^c u = c_d \det[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}]_{j,k} dV$$

where $dV = (\frac{i}{2})^d \sum_{j=1}^d dz_j \wedge d\bar{z}_j$ is the volume form on $\mathbb{C}^d$ and c_d is a dimensional constant. Here,

$$dd^c u = 2i \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k = \Delta u \cdot dV \text{ in } \mathbb{C}.$$

If u locally bounded psh $(dd^c u)^d$ well-defined as positive measure.

$P-$extremal functions and $P = \Sigma$ case

Using Hörmander $L^2 - \bar{\partial}-$ theory, T. Bayraktar showed:

Theorem

(T. Bayraktar) *We have*

$$V_{P,K}(z) = \lim_{n\to\infty} \left[\sup\{\frac{1}{n}\log|p_n(z)| : \ p_n \in Poly(nP), \ ||p_n||_K \leq 1\}\right].$$

This is the starting point to develop a $P-$pluripotential theory. For

$$P = \Sigma = \{(x_1, ..., x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \sum_{j=1}^{d} x_i \leq 1\},$$

$Poly(n\Sigma) = \mathcal{P}_n$, and we recover "classical" pluripotential theory:

$$H_\Sigma(z) = \max[0, \log|z_1|, ..., \log|z_d|] = \max[\log^+|z_1|, ..., \log^+|z_d|]$$

and $L_\Sigma = L$; $V_{\Sigma,K} = V_K$.

Remark: Global Domination Principle

We have $L_P^+ := \{u \in L_P : u(z) \geq H_P(z) + c_u\}$. We call $u_P \in PSH(\mathbb{C}^d)$ a *strictly psh $P-$potential* if

1. $u_P \in L_P^+$ is strictly psh and
2. there exists C such that $|u_P(z) - H_P(z)| \leq C$ for all $z \in \mathbb{C}^d$.

We have existence of u_P which may replace H_P:

$$L_P = \{u \in PSH(\mathbb{C}^d) : u(z) - u_P(z) = 0(1),\ |z| \to \infty\}$$

and

$$L_P^+ = \{u \in L_P : u(z) \geq u_P(z) + c_u\}.$$

Using u_P we can prove a global domination principle:

Theorem

Let $u \in L_P$ and $v \in L_P^+$ with $u \leq v$ a.e. $(dd^c v)^d$. Then $u \leq v$ in $\mathbb{C}^d$.

Recall $\mathbf{b_n}$ is the dimension of $Poly(nP)$. We can write

$$Poly(nP) = \text{span}\{e_1, ..., e_{\mathbf{b_n}}\}$$

where $\{e_j(z) := z^{\alpha(j)}\}_{j=1,...,\mathbf{b_n}}$ are the standard basis monomials. The ordering is unimportant. For points $\zeta_1, ..., \zeta_{\mathbf{b_n}} \in \mathbb{C}^d$, let

$$VDM_n^P(\zeta_1, ..., \zeta_{\mathbf{b_n}}) := \det[e_i(\zeta_j)]_{i,j=1,...,\mathbf{b_n}} \tag{1}$$

$$= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_{\mathbf{b_n}}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{\mathbf{b_n}}(\zeta_1) & e_{\mathbf{b_n}}(\zeta_2) & \ldots & e_{\mathbf{b_n}}(\zeta_{\mathbf{b_n}}) \end{bmatrix}.$$

For $K \subset \mathbb{C}^d$ compact, $P-$Fekete points of order n maximize $|VDM_n^P(\zeta_1, ..., \zeta_{\mathbf{b_n}})|$ over $\zeta_1, ..., \zeta_{\mathbf{b_n}} \in K$. Let $l_n := \sum_{j=1}^{\mathbf{b_n}} \deg(e_j)$. Then (nontrivial!) the limit

$$\delta(K, P) := \lim_{n\to\infty} \max_{\zeta_1,...,\zeta_{\mathbf{b_n}} \in K} |VDM_n^P(\zeta_1, ..., \zeta_{\mathbf{b_n}})|^{1/l_n} \text{ exists.}$$

Asymptotic $P-$Fekete arrays and $(dd^cV_{P,K})^d$

Theorem

Let $K \subset \mathbb{C}^d$ be compact. For each n, take points $z_1^{(n)}, z_2^{(n)}, \cdots, z_{\mathbf{b_n}}^{(n)} \in K$ for which

$$\lim_{n\to\infty} |VDM_n^P(z_1^{(n)}, \cdots, z_{\mathbf{b_n}}^{(n)})|^{\frac{1}{l_n}} = \delta(K,P)$$

(asymptotic $P-$Fekete arrays) and let $\mu_n := \frac{1}{\mathbf{b_n}} \sum_{j=1}^{\mathbf{b_n}} \delta_{z_j^{(n)}}$. Then

$$\mu_n \to \frac{1}{d!Vol(P)}(dd^cV_{P,K})^d \ weak-*.$$

The proof of these results rely on techniques from Berman and Boucksom, *Invent. Math.*, (2010) and involve *weighted* notions of $V_{P,K}$ and $\delta(K,P)$. See Bayraktar, Bloom, L., *Pluripotential Theory and Convex Bodies*. Remark: $(dd^cV_{P,K})^d$ is the "target" for zeros of random polynomial mappings – and our motivation.

Questions: product sets

Recall

$$V_{P,E_1\times\cdots\times E_d}(z_1,\ldots,z_d)=\phi_P(V_{E_1}(z_1),\ldots,V_{E_d}(z_d)).$$

For $T^d=\{(z_1,\ldots,z_d):|z_j|=1,\ j=1,\ldots,d\}$ and $1\le q\le\infty$, $(dd^cV_{P_q,T^d})^d$ is a multiple of Haar measure on T^d and for $[-1,1]^d$,

$$V_{P_q,[-1,1]^d}(z_1,\ldots,z_d)=\left\{\sum_{j=1}^{d}\left(\log\left|z_j+\sqrt{z_j^2-1}\right|\right)^{q'}\right\}^{1/q'}.$$

1. Is supp $(dd^cV_{P_q,[-1,1]^d})^d=[-1,1]^d$ *always*?
2. More generally, what can one say about supp$(dd^cV_{P_q,E_1\times\cdots\times E_d})^d$ for $1<q<\infty$?

Other sets: the complex Euclidean ball in $\mathbb{C}^2$

For $K = B_2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 \leq 1\}$ and $P = P_\infty$, we have shown:

$$V_{P_\infty, B_2}(z) = \begin{cases} \frac{1}{2}\left\{\log(|z_2|^2) - \log(1-|z_1|^2)\right\} & |z_1|^2 \leq 1/2,\ |z_2|^2 \geq 1/2 \\ \frac{1}{2}\left\{\log(|z_1|^2) - \log(1-|z_2|^2)\right\} & |z_1|^2 \geq 1/2,\ |z_2|^2 \leq 1/2 \\ \log|z_1| + \log|z_2| + \log(2) & |z_1|^2 \geq 1/2,\ |z_2|^2 \geq 1/2. \end{cases}$$

Thus the measure $(dd^c V^*_{P_\infty, B_2})^2$ is Haar measure on the torus $\{|z_1| = 1/\sqrt{2},\ |z_2| = 1/\sqrt{2}\}$ (with total mass 2). On the other hand, it is well known that $(dd^c V^*_{P_1, B_2})^2$ is normalized surface measure on ∂B_2. What can we say about $\text{supp}(dd^c V^*_{P_q, B_2})^2$ for $1 < q < \infty$? What *is* $V^*_{P_q, B_2}$?

A final question ... from numerical analysis

What can one say if P is not convex? For example, let

$$P_q := \{(x_1, ..., x_d) : x_1, ..., x_d \geq 0,\ x_1^q + \cdots + x_d^q \leq 1\},$$

for $0 < q < 1$. Is there a Bernstein-Walsh theorem? Even for "$q = 0$":

$$P_0 := \cup_{j=1}^d \{(x_1, ..., x_d) : 0 \leq x_j \leq 1,\ x_k = 0,\ k \neq j\}.$$

Let $K = [-1,1]^2 \subset \mathbb{R}^2 \subset \mathbb{C}^2$ and $f(z,w) = g(z) + h(w)$, g, h holomorphic in a neighborhood of $[-1,1]$. Here one should only use polynomials of the form $p(z) + q(w)$ and thus

$$D_n(f, P_q, K) = D_n(f, P_1, K),\ 0 \leq q \leq 1.$$

What is the "true" extremal function $V_{P,K}$?

CONGRATULATIONS
to TOM !!!

THANK YOU ALL !!!