The reachable space of the heat equation : a complex analysis approach

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We consider the the $1-d$ heat equation

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) & t \geq 0,\ x \in (0,\pi), \\ w(t,0) = u_0(t),\ \ w(t,\pi) = u_\pi(t) & t \in [0,\infty), \\ w(0,x) = 0 & x \in (0,\pi), \end{cases} \tag{1}$$

Given $\tau > 0$, define the input to state map $(\Phi_\tau)_{\tau>0}$ by

$$\Phi_\tau \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix} = w(\tau,\cdot) \qquad (\tau > 0,\ u_0,\ u_\pi \in L^2[0,\tau]),$$

Problem

Describe the reachable set

$$\operatorname{Ran}\Phi_\tau = \left\{ w(\tau,\cdot) : w \text{ solving the heat equation with } u_0,\ u_\pi \in L^2[0,\tau] \right\}$$

$$\operatorname{Ran}\Phi_\tau = \left\{ w(\tau,x) = \sum_{n\geqslant 1} n \left(\int_0^\tau e^{n^2(\sigma-\tau)} (u_0(\sigma) + (-1)^n u_\pi(\sigma))\, d\sigma \right) \sin(nx), \quad x \in (0,\pi)) \right\}$$

Known results

- Fattorini, Seidman $\operatorname{Ran}\Phi_\tau$ does not depend on time
- $\operatorname{Ran}\Phi_\tau \subset \operatorname{Hol}(D)$ where

$$D = \{s = x + iy \in \mathbb{C} : |y| < x \text{ and } |y| < \pi - x\}$$

- Fattorini–Russel 1971

$$\left\{\psi(x) = \sum c_n \sin(nx) \quad x \in [0,\pi] : \sum_n |c_n|^2 n e^{n\pi} < \infty\right\} \subset \operatorname{Ran}\Phi_\tau$$

- Ervedoza-Zuazua 2012

$$\left\{\psi \in \operatorname{Hol}(S) : \psi^{(2k)}(0) = \psi^{(2k)}(\pi) = 0\right\} \subset \operatorname{Ran}\Phi_\tau$$

  where

$$S = \{s = x + iy \in \mathbb{C} : |y| < \pi\}.$$

- Martin-Rossier-Rouchon 2015

$$\operatorname{Hol}(B) \subset \operatorname{Ran}\Phi_\tau$$

  where

$$B = \{s \in \mathbb{C} : |s - \frac{\pi}{2}| < \frac{\pi}{2} e^{(2e)^{-1}}\}$$

- Dardé-Ervedoza 2016

$$\operatorname{Hol}(D_\varepsilon) \subset \operatorname{Ran}\Phi_\tau$$

  where D_ε an ε–neighbourhood of D .

Main results

Theorem

For every $\tau > 0$ *we have*

$$E^2(D) \subsetneq \operatorname{Ran}\Phi_\tau \subset A^2(D)$$

Fourier series expansion

Writting the solution $\Phi_\tau \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix}(x) = w(t,x) = \sum_{n\geq 1} w_n(t)\sin(nx)$, we obtain

$$\Phi_\tau \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix}(x) = \frac{2}{\pi}\sum_{n\geqslant 1} n\left[\int_0^\tau e^{n^2(\sigma-\tau)}u_0(\sigma)\,d\sigma\right]\sin(nx)$$
$$+\frac{2}{\pi}\sum_{n\geqslant 1} n(-1)^{n+1}\left[\int_0^\tau e^{n^2(\sigma-\tau)}u_\pi(\sigma)\,d\sigma\right]\sin(nx) \qquad (\tau>0,\ x\in(0,\pi)),$$

Lemma

$$\Phi_\tau \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix}(x) = \int_0^\tau \frac{\partial K_0}{\partial x}(\tau-\sigma,x)u_0(\sigma)\,d\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau-\sigma,x)u_\pi(\sigma)\,d\sigma,$$

where

$$K_0(\sigma,x) = -\sqrt{\frac{1}{\pi\sigma}}\sum_{m\in\mathbb{Z}} e^{-\frac{(x+2m\pi)^2}{4\sigma}} \quad \textit{and} \quad K_\pi(\sigma,x) = -K_0(\sigma,\pi-x) \qquad (\sigma>0, x\in\mathbb{R}),$$

Proof.

Apply the Poisson summation formula in the Fourier series representation □

Proof of the Proposition : $\operatorname{Ran}\Phi_\tau \subset A^2(D)$

$$\Phi_\tau\begin{pmatrix}u_0\\ u_\pi\end{pmatrix}(x) = \int_0^\tau \frac{\partial K_0}{\partial x}(\tau-\sigma,x)u_0(\sigma)\,d\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau-\sigma,x)u_\pi(\sigma)\,d\sigma,$$

where

$$K_0(\sigma,x) = -\sqrt{\frac{1}{\pi\sigma}}\,e^{-\frac{x^2}{4\sigma}} - \underbrace{\sqrt{\frac{1}{\pi\sigma}}\sum_{m\neq 0} e^{-\frac{(x+2m\pi)^2}{4\sigma}}}_{\widetilde{K}_0(\sigma,x)} \quad \text{and} \quad K_\pi(\sigma,x) = -K_0(\sigma,x-\pi)$$

$$\Phi_\tau\begin{pmatrix}u_0\\ u_\pi\end{pmatrix}(x) = (\phi_\tau^0 u)(x) + \int_0^\tau \frac{\partial \widetilde{K}_0}{\partial x}(\tau-\sigma,x)u_0(\sigma)\,d\sigma + (\phi_\tau^\pi u)(x) + \int_0^\tau \frac{\partial \widetilde{K}_\pi}{\partial x}(\tau-\sigma,x)u_\pi(\sigma)\,d\sigma$$

where

$$(\phi_\tau^0 u)(s) := \frac{1}{2\sqrt{\pi}}\int_0^\tau \frac{s e^{-\frac{s^2}{4(\tau-\sigma)}}}{(\tau-\sigma)^{\frac{3}{2}}} f(\sigma)\sigma\,d\sigma \quad \text{and} \quad (\phi_\tau^\pi u)(s) = (\phi_\tau^0 u)(\pi - s)$$

Since there exist $a,\ b>0$ such that for every $k\in\mathbb{Z}\setminus\{-1,0\}$ we have

$$\left|(s+k\pi)e^{-\frac{(s+k\pi)^2}{4(\tau-\sigma)}}\right|^2 \leqslant ak^2 e^{\frac{-bk^2}{(\tau-\sigma)}} \qquad (s\in D).$$

It suffices to show that $\phi_\tau^0 u$ and $\phi_\tau^\pi u$ can be be extended to a function in $A^2(D)$

A result of Aikawa, Hayashi and Saitoh

Let

$$\Delta = \left\{ s \in \mathbb{C} \;\middle|\; -\frac{\pi}{4} < \arg s < \frac{\pi}{4} \right\}.$$

Let ω be a positive measurable function on Δ, then

$$A^2(\Delta, \omega) = \left\{ f \in \mathrm{Hol}(D) : \int_D |f(x+iy)|^2 \omega(x+iy)\,dxdy < \infty \right\}$$

Theorem

For $s \in \Delta$, $\tau > 0$ and $f \in L^2[0,\tau]$ we set

$$(P_\tau f)(s) = \frac{1}{2\sqrt{\pi}} \int_0^\tau \frac{s e^{-\frac{s^2}{4(\tau-\sigma)}}}{(\tau-\sigma)^{\frac{3}{2}}} f(\sigma)\sqrt{\sigma}\,d\sigma.$$

Then P_τ defines an isometric isomorphism from $L^2[0,\tau]$ onto $A^2(\Delta,\omega_0)$, where $\omega_0(s) = \frac{e^{\frac{\mathrm{Re}(s^2)}{2\tau}}}{\tau}$.

Corollary

$\phi_\tau^0 \in \mathcal{L}(L^2[0,\tau], A^2(D))$

A resul of Levin-Lyubarski

Let $\diamond : \{s : x+iy \in \mathbb{C} : |x| \leq \pi/2 \quad \text{and} \quad |y| \leq \pi/2\}$

$$H(\lambda) = \frac{\pi}{2}\begin{cases} \operatorname{Im}\lambda & \text{if } \arg\lambda \in (\pi/4, 3\pi/4) \\ -\operatorname{Re}\lambda & \text{if } \arg\lambda \in (3\pi/4, 5\pi/4), \\ -\operatorname{Im}\lambda & \text{if } \arg\lambda \in (5\pi/4, 7\pi/4, \\ \operatorname{Re}\lambda & \text{if } \arg\lambda \in (5\pi/4, \pi/4). \end{cases}$$

We say that an entire function S belongs to the class $\mathcal{S}_\diamond$ if

1. The zero set Λ of S is separated : there exists $\delta > 0$ such that
$$\inf\{|\lambda - \lambda^*| : \lambda, \lambda^* \in \Lambda\} \geq \delta > 0.$$
2. There exists $K > 0$ such that
$$\operatorname{dist}(\lambda, \mathrm{e}^{i\frac{i\pi}{4}}\mathbb{R} \times \mathrm{e}^{-i\frac{\pi}{4}}\mathbb{R}) < K.$$
3. There exists $A > 1$ such that
$$\frac{1}{A}\, e^{H(z)} \operatorname{dist}(z, \Lambda) \leq |S(z)| \leq A\, e^{H(z)} \operatorname{dist}(z, \Lambda).$$

A resul of Levin-Lyubarski

Theorem

Let Λ the zero set of $S \in \mathcal{S}_\diamond$, the family $(e_\lambda)_{\lambda\in\Lambda}$ is a Riesz basis in $E^2(\diamond)$ where

$$e_\lambda(s) = e^{\lambda s} e^{-H(\lambda)}, \qquad \lambda \in \Lambda,\ s \in \diamond.$$

The family $(e_\lambda)_{\lambda\in\Lambda}$ is a Riesz basis for $E^2(D)$, if and only if, the operator

$$\begin{array}{cccc} T_\Lambda : & E^2(D) & \longrightarrow & \ell^2(\Lambda) \\ & f & \longmapsto & \left(\langle f, e_\lambda\rangle_{E^2(D)}\right)_{\lambda\in\Lambda} \end{array}$$

is bounded and invertible from $E^2(D)$ to $\ell^2(\Lambda)$.

Corollary

Let $\Lambda = \{(2k+1)(1\pm i) : k \in \mathbb{Z}\}$. The family $(e_\lambda)_{\lambda\in\Lambda}$ is a Riesz basis in $E^2(D)$ where

$$e_\lambda(s) = e^{\lambda(s-\frac{\pi}{2})} e^{-H(\lambda)}, \qquad \lambda \in \Lambda,\ s \in \mathbb{C}.$$

Lemma

let $\tau > 0$ *and* $\varphi \in E^2(D)$*. Then there exists* $\varphi_1 \in A^2(\Delta, \omega_0)$ *and* $\varphi_2 \in A^2(\pi - \Delta, \omega_\pi)$ *such that*

$$\varphi(s) = \varphi_1(s) + \varphi_2(s) \qquad (s \in D).$$

here $\omega_\pi(s) = \omega_0(\pi - s)$*, for* $s \in \pi - \Delta$*.*

Proof.

Let $\varphi \in E^2(D)$, $\varphi = \sum_{\lambda \in \Lambda} a_\lambda e_\lambda$

$$\varphi_1(s) = F(s) \sum_{\substack{\lambda \in \Lambda \\ \mathrm{Re}\,\lambda < 0}} a_\lambda e_\lambda(s) \qquad s \in \Delta$$

$$\varphi_2(s) = F(\pi - s) \sum_{\substack{\lambda \in \Lambda \\ \mathrm{Re}\,\lambda > 0}} a_\lambda e_\lambda(s), \qquad s \in \pi - \Delta$$

we use Hilbert's inequality to prove required estimates. □

Proof of the main result

Let

$$\widetilde{K}_0(\sigma,x) = -\sqrt{\frac{1}{\pi\sigma}}\sum_{m\neq 0} e^{-\frac{(x+2m\pi)^2}{4\sigma}} \quad \text{and} \quad \widetilde{K}_\pi(\sigma,x) = \widetilde{K}_0(\sigma,\pi - x)$$

We decompose these functions as

$$\tilde{K}_0(\sigma,s) = A(\sigma,s) + B(\sigma,s), \quad \text{and} \quad \tilde{K}_\pi(\sigma,s) = C(\sigma,s) + D(\sigma,s),$$

where

$$A(\sigma,s) = -\sqrt{\frac{1}{\pi\sigma}}\sum_{m\geqslant 1} e^{-\frac{(s+2m\pi)^2}{4\sigma}} \qquad \text{and} \qquad C(\sigma,s) = A(\sigma,\pi - s)$$

Denote

$$R_{A,\tau}u(s) = \int_0^\tau \frac{\partial A}{\partial s}(\tau-\sigma,s)\sqrt{\sigma}u(\sigma)d\sigma,$$

and similarly introduce $R_{B,\tau}$, $R_{C,\tau}$ and $R_{D,\tau}$.

$$\Phi_\tau \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix} = \begin{pmatrix} P_\tau + R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & Q_\tau + R_{D,\tau} \end{pmatrix} \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix}$$

where $Q_\tau f(s) = P_\tau f(\pi - s)$ for $s \in \pi - \Delta$

Proof of the main result

Lemma

$$\lim_{\tau\to 0+}\left\|\begin{pmatrix} R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & R_{D,\tau}\end{pmatrix}\right\|_{\mathcal{L}\left(L^2[0,\pi]\times L^2[0,\pi],A^2(\Delta,\omega_0)\times A^2(\pi-\Delta,\omega_\pi)\right)} = 0\,.$$

$$\Phi_\tau = \begin{pmatrix} P_\tau + R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & Q_\tau + R_{D,\tau}\end{pmatrix} \in \mathcal{L}\left(L^2[0,\pi]\times L^2[0,\pi], A^2(\Delta,\omega_0)\times A^2(\pi-\Delta,\omega_\pi)\right).$$

Since

$$\begin{pmatrix} P_\tau & 0 \\ 0 & Q_\tau\end{pmatrix} \in \mathcal{L}\left(L^2[0,\pi]\times L^2[0,\pi], A^2(\Delta,\omega_0)\times A^2(\pi-\Delta,\omega_\pi)\right)$$

is invertible

$$\left\|\begin{pmatrix} P_\tau & 0 \\ 0 & Q_\tau\end{pmatrix}\right\| = 1 \quad \text{and} \quad \lim_{\tau\to 0+}\left\|\begin{pmatrix} R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & R_{D,\tau}\end{pmatrix}\right\| = 0\,.$$

Φ_τ is invertible for τ small.

Let $\varphi \in E^2(D)$. Then there exists a decomposition

$$\varphi(s) = \varphi_1(s) + \varphi_2(s) \qquad (s \in D).$$

where $\varphi_1 \in A^2(\Delta, \omega_0)$ and $\varphi_2 \in A^2(\pi - \Delta, \omega_\pi)$. Since ϕ_{τ^*} is invertible for some small τ^* there exists $u_0, u_\pi \in L^2[0, \tau^*]$ such that

$$\phi_{\tau^*} \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

So $E^2(D) \subset \mathrm{Ran}\,\Phi_{\tau^*}$ and $\mathrm{Ran}\,\Phi_\tau$ is independent of $\tau > 0$.

Newmann control

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) & t \geq 0,\ x \in (0,\pi), \\ \dfrac{\partial w}{\partial x}(t,0) = u_0(t),\ \ \dfrac{\partial w}{\partial x}(t,\pi) = u_\pi(t) & t \in [0,\infty), \\ w(0,x) = 0 & x \in (0,\pi). \end{cases}$$

Given $\tau > 0$, the input to state map is

$$\Phi_\tau^N \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix} = w(\tau,\cdot) \qquad (\tau > 0,\ u_0,\ u_\pi \in L^2[0,\tau]),$$

Corollary

For every $\tau > 0$ we have

$$E^{2,1}(D) \subsetneq \mathrm{Ran}\,\Phi_\tau^N \subset \mathcal{D}(D),$$

where

$$E^{2,1}(D) = \{f \in \mathrm{Hol}(D) : f' \in E^2(D)\}$$

$$\mathcal{D}(D) = \{f \in \mathrm{Hol}(D) : f' \in A^2(D)\}$$