# Spectral function, remainder in Weyl's law and resonances: a survey

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**Spectral function, Weyl's law:**

- [JP]: GAFA, 17 (2007), 806-838.
- [JPT]: IMRN Volume 2007: article ID rnm142.
- [DJ]: Journal of Modern Dynamics 10 (2016), 339-352.

$X^n, n \geq 2$ - compact. $\Delta$ - Laplacian. Spectrum:
$\Delta\phi_i + \lambda_i\phi_i = 0, \;\; 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$

**Eigenvalue counting function:**
$N(\lambda) = \#\{\sqrt{\lambda_j} \leq \lambda\}$.

**Weyl's law:** $N(\lambda) = C_n V \lambda^n + R(\lambda), \; R(\lambda) = O(\lambda^{n-1})$ (Avakumovic, Levitan; Hörmander - more general operators). $R(\lambda)$ - **remainder.** Duistermaat-Guillemin: $R(\lambda) = o(\lambda^{n-1})$ if the set of periodic geodesics in $T^1X$ (unit sphere bundle of $X$) has measure 0. On $S^n$, all geodesics are periodic (also on Zoll manifolds); $R(\lambda) \asymp \lambda^{n-1}$ on $S^n$.

**Manifolds with boundary.** H. Weyl conjectured (1913) that for the Laplacian in a domain $\Omega$ of dimension $n$, we have

$$N(\lambda) = c_0 \mathrm{vol}_n(\Omega)\lambda^n \mp c_1 \mathrm{vol}_{n-1}(\partial\Omega)\lambda^{n-1} + o\left(\lambda^{n-1}\right). \qquad (1)$$

Here $-$ corresponds to Dirichlet, and $+$ to Neumann boundary conditions, and $c_j$-s depend only on $n$.
Courant (1920): $R(\lambda) = O(\lambda^{n-1}\log\lambda)$.
Seeley (1978): $R(\lambda) = O(\lambda^{n-1})$.
Ivrii, Melrose: proved H. Weyl's conjecture (1), provided *the set of periodic billiard trajectories in* $\Omega$ *has measure* 0. Ivrii conjectured that this condition holds for *general* Euclidean domains; that conjecture is still open.

**Spectral function:** Let $x, y \in X$.
$N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \le \lambda} \phi_i(x)\phi_i(y)$.
If $x = y$, let $N_{x,y}(\lambda) := N_x(\lambda)$.
**Local Weyl's law:**
$N_{x,y}(\lambda) = O(\lambda^{n-1}), \qquad x \neq y;$
$N_x(\lambda) = C_n\lambda^n + R_x(\lambda), \qquad R_x(\lambda) = O(\lambda^{n-1})$; $R_x(\lambda)$ - **local remainder**.
We shall discuss **lower** bounds for $R(\lambda), R_x(\lambda)$ and $N_{x,y}(\lambda)$.
Notation: $f_1(\lambda) = \Omega(f_2(\lambda)), f_2 > 0$ iff
$\limsup_{\lambda\to\infty} |f_1(\lambda)|/f_2(\lambda) > 0$. Equivalently, $f_1(\lambda) \neq o(f_2(\lambda))$.

**Lower bounds:**
**Theorem 1**[JP] If $x, y \in X$ are not conjugate along any shortest geodesic joining them, then

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right).$$

**Theorem 2**[JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then

$$R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

Other results in dimension $n > 2$ involve heat invariants.

**Example: flat square 2-torus**
$\lambda_j = 4\pi^2(n_1^2 + n_2^2), \quad n_1, n_2 \in \mathbf{Z}$
$\phi_j(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)}, \; x = (x_1, x_2)$

$$|\phi_j(x)| = 1 \;\Rightarrow\; N(\lambda) \equiv N_x(\lambda)$$

**Gauss circle problem:** estimate $R(\lambda)$.
Theorem 2 $\Rightarrow$ $\qquad R(\lambda) = \Omega(\sqrt{\lambda})$ -
**Hardy–Landau bound.** Theorem 2 generalizes that bound for the *local* remainder.

**Soundararajan** (2003): $R(\lambda) = \Omega\left(\frac{\sqrt{\lambda}(\log\lambda)^{\frac{1}{4}}(\log\log\lambda)^{\frac{3(2^{4/3}-1)}{4}}}{(\log\log\log\lambda)^{5/8}}\right)$.

**Hardy's conjecture:** $R(\lambda) \ll \lambda^{1/2+\epsilon}\ \forall\epsilon > 0$.
**Huxley** (2003): $R(\lambda) \ll \lambda^{\frac{131}{208}}(\log\lambda)^{2.26}$.

**Negative curvature.** Suppose sectional curvature satisfies $-K_1^2 \leq K(\xi, \eta) \leq -K_2^2$
**Theorem (Berard)**: $R_x(\lambda) = O(\lambda^{n-1}/\log\lambda)$
**Conjecture (Randol)**: On a negatively-curved surface, $R(\lambda) = O(\lambda^{\frac{1}{2}+\epsilon})$. Randol proved an integrated (in $\lambda$) version for $N_{x,y}(\lambda)$.
**Theorem (Karnaukh)** On a negatively curved surface

$$R_x(\lambda) = \Omega(\sqrt{\lambda})$$

+ logarithmic improvements discussed below. Karnaukh's results (unpublished 1996 Princeton Ph.D. thesis under the supervision of P. Sarnak) served as a starting point and a motivation for our work.

**Thermodynamic formalism:** $G^t$ - geodesic flow on $SX$, $\xi \in SX$. **Sinai-Ruelle-Bowen potential** (= "unstable jacobian") $\mathcal{H} : SX \to \mathbb{R}$:

$$\mathcal{H}(\xi) = \left.\frac{d}{dt}\right|_{t=0} \ln \det dG^t|_{E^u_\xi}$$

where dim $E^u_\xi \subset T_\xi(SX)$ is the *unstable subspace* exponentially contracting for the *inverse* flow $G^{-t}$ (flow-invariant, dim $= n-1$.)
**Topological pressure** $P(f)$ of a Hölder function $f : SX \to \mathbb{R}$ satisfies (Parry, Pollicott)

$$\sum_{l(\gamma)\leq T} l(\gamma) \exp\left[\int_\gamma f(\gamma(s), \gamma'(s))ds\right] \sim \frac{e^{P(f)T}}{P(f)}.$$

$\gamma$ - geodesic of length $l(\gamma)$.

**Examples:**
(a): $P(0) = h$ - **topological entropy** of $G^t$. Theorem (Margulis): $\#\{\gamma : l(\gamma) \leq T\} \sim e^{hT}/hT$.
(b): $P(-\mathcal{H}/2) \geq (n-1)K_2/2$
(c): $P(-\mathcal{H}) = 0$.

**Theorem 3**[JP] If $X$ is negatively-curved then for any $\delta > 0$ and $x \neq y$

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right)$$

Since $h \leq (n-1)K_1$, we have $P(-\mathcal{H}/2)/h \geq K_2/(2K_1) > 0$.

**Theorem 4a**[JP] $X$ - negatively-curved. For any $\delta > 0$

$$R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right),\ n = 2, 3.$$

Results for $n \geq 4$ : need to subtract heat invariants.

$$K = -1\ \Rightarrow R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log\lambda)^{\frac{1}{2} - \delta}\right)$$

**Karnaukh,** $n = 2$: estimate above + weaker estimates in variable negative curvature.

**Global results:** $R(\lambda)$
**Randol,** $n = 2$**:**

$$K = -1 \;\Rightarrow R(\lambda) = \Omega\left((\log\lambda)^{\frac{1}{2}-\delta}\right), \qquad \forall\delta > 0.$$

**Theorem 4b**[JPT] $X$ - negatively-curved surface ($n = 2$). For any $\delta > 0$

$$R(\lambda) = \Omega\left((\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right).$$

**Conjecture (folklore).** On a **generic** negatively curved surface

$$R(\lambda) = O(\lambda^{\epsilon}) \qquad \forall\epsilon > 0.$$

**Selberg, Hejhal:** On general compact hyperbolic surfaces,

$$R(\lambda) = \Omega\left(\frac{(\log\lambda)^{\frac{1}{2}}}{\sqrt{\log\log\lambda}}\right).$$

On compact arithmetic surfaces that correspond to quaternionic lattices $R(\lambda) = \Omega\left(\frac{\sqrt{\lambda}}{\log\lambda}\right)$.
**Reason:** *exponentially high* multiplicities in the length spectrum; generically, $X$ has *simple* length spectrum.

[JN10]: similar ideas are used to obtain lower bounds for resonances of infinite area hyperbolic surfaces.

**Proof of Theorem 4b:** (about $R(\lambda)$). $X$-compact, negatively-curved surface.
**Wave trace** on $X$ (even part):

$$e(t) \; = \; \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i} t).$$

**Cut-off:** $\chi(t, T) = (1 - \psi(t))\hat{\rho}\left(\frac{t}{T}\right)$, where
• $\rho \in \mathcal{S}(\mathbb{R})$, $\mathrm{supp}\,\hat{\rho} \subset [-1, +1]$, $\rho \geq 0$, even;
• $\psi(t) \in C_0^\infty(\mathbb{R})$, $\psi(t) \equiv 1, t \in [-T_0, T_0]$, and $\psi(t) \equiv 0, |t| \geq 2T_0$.
In the sequel, $T = T(\lambda) \to \infty$ as $\lambda \to \infty$. Let

$$\kappa(\lambda, T) = \frac{1}{T} \int_{-\infty}^{\infty} e(t)\chi(t, T) \cos(\lambda t) dt$$

**Key microlocal result**:
**Proposition 9.** Let $T = T(\lambda) \leq \epsilon \log \lambda$. Then

$$\kappa(\lambda, T) = \sum_{l(\gamma) \leq T} \frac{l(\gamma)^{\#} \cos(\lambda l(\gamma)) \cdot \chi(l(\gamma), T)}{T\sqrt{|\det(I - \mathcal{P}_\gamma)|}} + O(1)$$

where
$\gamma$ - closed geodesic; $l(\gamma)$ - length; $l(\gamma)^{\#}$-primitive period; $\mathcal{P}_\gamma$ - Poincaré map.
*Long-time* version of the “wave trace” formula of Duistermaat and Guillemin, microlocalized to shrinking neighborhoods of closed geodesics. Allows to isolate contribution from a **growing number** of closed geodesics with $l(\gamma) \leq T(\lambda)$ to $\kappa(\lambda, T)$ as $\lambda, T(\lambda) \to \infty$.

**Proof** - separation of closed geodesics in phase space + small-scale microlocalization near closed geodesics.
**Dynamical lemma**: Let $X$ - compact, negatively curved manifold. $\Omega(\gamma, r)$ - neighborhood of $\gamma$ in $S^*X$ of radius $r$ (cylinder). There exist constants $B > 0, a > 0$ s.t. for all closed geodesics on $X$ with $l(\gamma) \in [T - a, T]$, the neighborhoods $\Omega(\gamma, e^{-BT})$ are disjoint, provided $T > T_0$.
Radius $r = e^{-BT}$ is exponentially small in $T$, since the number of closed geodesic grows exponentially.
**Remark:** on a dense set of negatively curved metrics, there is no exponential lower bound between lengths of different closed geodesics ([DJ]), so it is essential to work in phase space, where the Dynamical lemma provides the required separation. However, such separation holds for many *hyperbolic* manifolds, since the generators of $\pi_1(X)$ are matrices with *algebraic* entries.

Our **local** estimates are not uniform in $x, y$. Need Proposition 9 to prove **global** estimates.

**Heat trace asymptotics:**

$$\sum_i e^{-\lambda_i t} \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j \; t^{j-\frac{n}{2}}, \qquad t \to 0^+$$

**Local:** $\mathcal{K}(t, x, x) = \sum_i e^{-\lambda_i t} \phi_i^2(x) \sim \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) t^{j-\frac{n}{2}},$

$a_j(x)$ **- local heat invariants,** $a_j = \int_X a_j(x) dx$**.**

$a_0(x) = 1$**,** $a_0 = \mathrm{vol}(X)$**.** $a_1(x) = \frac{\tau(x)}{6}$**,** $\tau(x)$ **- scalar curvature.**

**“Heat kernel” estimates:**

**Theorem 2b**[JP] If the scalar curvature $\tau(x) \neq 0, \Longrightarrow R_x(\lambda) = \Omega(\lambda^{n-2})$.

**Global:**[JPT] If $\int_X \tau \neq 0, \Rightarrow R(\lambda) = \Omega(\lambda^{n-2})$.

**Remark:** if $\tau(x) = 0$, let $k = k(x)$ be the first positive number such that the $k$-th local heat invariant $a_k(x) \neq 0$. If $n - 2k(x) > 0$, then

$$R_x(\lambda) = \Omega(\lambda^{n-2k(x)}).$$

Similar result holds for $R(\lambda)$: if $\int a_k(x)dx \neq 0$ and $n - 2k > 0$, then

$$R(\lambda) = \Omega(\lambda^{n-2k}).$$

**Oscillatory error term:** subtract $[(n-1)/2]$ terms coming from the heat trace:

$N_x(\lambda) = \sum_{j=0}^{[\frac{n-1}{2}]} \frac{a_j(x)\lambda^{n-2j}}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}-j+1\right)} + R_x^{osc}(\lambda)$

*Warning*: **not** an asymptotic expansion!

Physicists: subtract the "mean smooth part" of $N_x(\lambda)$.

**Theorem 2c**[JP] If $x \in X$ is not conjugate to itself along any shortest geodesic loop, then

$$R_x^{osc}(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

**Theorem 4c**[JP] $X$ - negatively-curved. For any $\delta > 0$

$R_x^{osc}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right),$ *any n*.

If $n \geq 4$ then Theorem 2b, $R_x(\lambda) = \Omega(\lambda^{n-2})$ gives a better bound for $R_x(\lambda)$.

**Global Conjecture:** $X$ - negatively-curved. For any $\delta > 0$

$R^{osc}(\lambda) = \Omega\left((\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right),$ *any n*.

**Almost periodic properties of the remainder:**
Let $N(R)$ be the eigenvalue counting function on $\mathbb{T}^2$.
Heath-Brown in 1990s showed that the quantity

$$f(R) = (N(R) - \pi R^2)/\sqrt{R},$$

has a limiting distribution, i.e. that

$$\lim_{T\to\infty} \frac{\mathrm{meas}\{R \in [T, 2T] : f(R) \in [a, b]\}}{T} = \int_a^b P(s)ds,$$

where $P(s)$ is a.c. w.r. to the Lebesgue measure, and $|P(s)| \ll \exp(-|s|^4)$ as $|s| \to \infty$.
Many related results for a shifted circle problem, as well as for the remainder term on surfaces of revolution, Zoll surfaces, and Liouville tori (metric has the form $(f(x) + g(y))(dx^2 + dy^2)$) were obtained by Bleher, Dyson, Lebowitz, Minasov, Kosygin, Sinai et al. They showed that $f(R)$ is a $B^2$ almost periodic function in the sense of Besikovitch.

Almost periodic properties of the remainder may be a more general phenomenon. For e.g. negatively-curved manifolds, Aurich, Bolte and Steiner *conjectured* that a (suitably normalized) remainder has a limiting distribution that is Gaussian. There are very few rigorous results in this direction. The behavior of $N(x, y, \lambda)/(\lambda^{(n-1)/2})$ was studied by Lapointe, Polterovich and Safarov. They showed the following. Let $N_E(\lambda, d) = (2\pi)^{-n/2} d^{-n/2} \lambda^{n/2} J_n(d\lambda)$ be the "Euclidean" spectral function; here $d = d(x, y)$. Then there exists $C_M$ s.t.

$$\int_M \left| \frac{N(x, y, \lambda) - N_E(\lambda, d(x, y))}{\lambda^{(n-1)/2}} \right|^2 \, dV(y) \leq C_M$$

Also, for any finite measure $d\nu(\lambda)$ on $\mathbb{R}$, and for any fixed $x \in M$, there exists $M_{x,\nu} \subset M$ of the full measure s.t. $\forall y \in M_{x,\nu}$,

$$\int_0^\infty |N(x, y, \lambda)|^2 / \lambda^{n-1} d\nu(\lambda) < \infty.$$

Further related results were obtained by B. Khanin and Y. Canzani; they studied $N(x, y, \lambda)$ near the "diagonal" $x = y$.

**Resonances**

Joint work with F. Naud, [JNS17] also with L. Soares.

[JN10] *Lower bounds for resonances of infinite area Riemann surfaces*. Journal of Analysis and PDE, vol. 3 (2010), no. 2, 207-225.

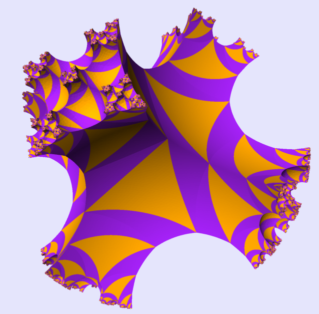
[JN12] *On the critical line of convex co-compact hyperbolic surfaces.* GAFA vol. 22, no. 2 (2012), 352-368.

[JN16] *Resonances and density bounds for convex co-compact congruence subgroups of $SL_2(\mathbb{Z})$.* Israel Jour. of Math. 2016, vol. 213 (2016), 443-473.

[JNS17] *Large covers and sharp resonances of hyperbolic surfaces.* arxiv:1710.05666

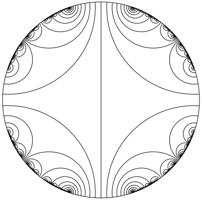
# *Hyperbolic manifolds*

Let $\mathbb{H}^{n+1}$ be the usual real hyperbolic space with sectional curvatures $-1$ and $\Gamma$ a *discrete* group of isometries.
The orbit of any $z \in \mathbb{H}^{n+1}$ under $\Gamma$-action accumulates at infinity, the limit set is defined by $\Lambda(\Gamma) := \partial\mathbb{H}^{n+1} \cap \overline{\Gamma.z}$.
A discrete group $\Gamma$, without elliptic elements, is called *convex co-compact* iff the convex hull of the limit set is co-compact.



The quotient $X = \Gamma\backslash\mathbb{H}^{n+1}$ is an *infinite volume* hyperbolic manifold called *convex co-compact.*
Let $\delta = \delta(\Gamma)$ be the Hausdorff dimension of the limit set.
The *geodesic flow* on $\phi_t : SX \to SX$ has a maximal compact invariant subset $\mathcal{T} \subset SX$, the *"trapped set"* with dimension $2\delta + 1$,

$$\mathcal{T} \simeq (\Lambda \times \Lambda \setminus D) \times \mathbb{R} \bmod \Gamma.$$



Liouville-almost all orbits of $\phi_t$ escape to infinity i.e. $Vol(\mathcal{T}) = 0$.

# *Laplace spectrum and resonances*

Let $\Delta_X$ be the hyperbolic *Laplacian* on $X = \Gamma\backslash\mathbb{H}^{n+1}$.
Lax-Phillips classical results describe the $L^2$-spectrum of $\Delta_X$:

1. Point spectrum is finite and in $(0, n^2/4)$.
2. Absolutly continuous spectrum is $[n^2/4, +\infty)$.
3. No embbeded eigenvalues in $[n^2/4, +\infty)$.

If $\delta > n/2$, then $\delta(n-\delta)$ is an eigenvalue, it's the bottom of the spectrum.
If $\delta \leq n/2$, point spectrum is empty.
Let

$$R(s) := (\Delta_X - s(n-s))^{-1} : L^2(X) \to L^2(X)$$

be the *resolvent*, meromorphic on the physical sheet $\{\Re(s) > n/2\}$.

From *[Mazzeo-Melrose, 1987]* we know that

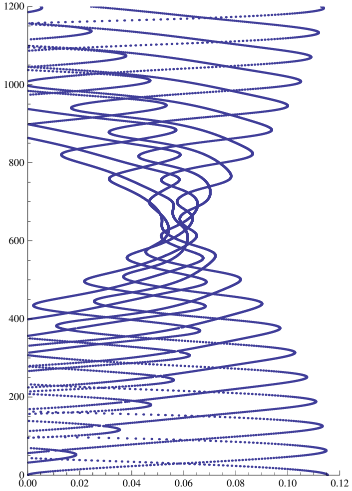
$$R(s) : C_0^\infty(X) \to C^\infty(X)$$

continues *meromorphically* to $\mathbb{C}$. Poles are called *resonances*. The structure of $R(s)$ at a resonance $s_0$ is given by a finite *Laurent* expansion

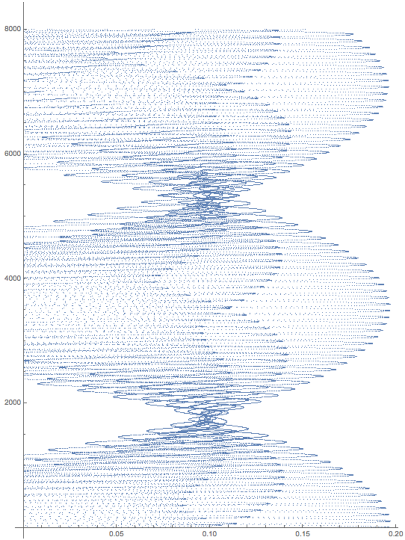
$$R(s) = \sum_j \frac{A_j(s_0)}{(s(n-s) - s_0(n-s_0))^j} + \textit{Holomorphic},$$

where $A_1(s_0) = \sum_{k=1}^{mult(s_0)} \phi_k \otimes \phi_k$, with $\Delta\phi_k = s_0(n-s_0)\phi_k$. Each (non-$L^2$) eigenfunction $\phi_k$ is called a *resonant state*.

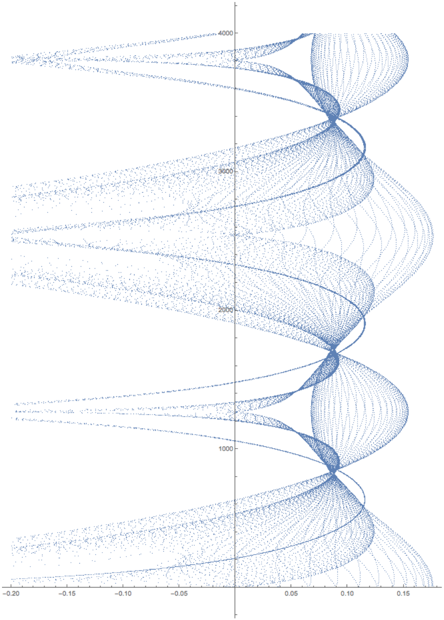
# Numerical computations for surfaces




Numerics by D. Borthwick (2012)



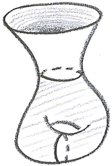
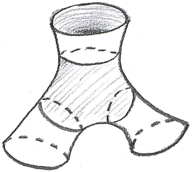

*Numerics by D. Borthwick & T. Weich (2017)*



4000
3000
2000
1000
-0.20
-0.15
-0.10
-0.05
0.05
0.10
0.15

# *First resonance*

*From now on*, we set $n = 2$, so that $X = \Gamma\backslash\mathbb{H}^2$ is a surface.

**Theorem** [Patterson: 1976, 1989]. For a convex co-compact quotient $X = \Gamma\backslash\mathbb{H}^2$, the first resonance is a simple pole at $s = \delta$, with no other resonances on $\{\Re(s) = \delta\}$, in particular $\Delta_X$ has eigenvalues iff $\delta > \frac{1}{2}$.

**Theorem** [Ballmann-Matthiesen-Mondal: 2016]. For a convex co-compact quotient $X = \Gamma\backslash\mathbb{H}^2$, the number of eigenvalues is at most $-\chi(X)$.

Proof generalizes results of Otal-Rosas for small eigenvalues on compact hyperbolic surfaces. Higher dimensions??

# *Global upper bounds*

**Theorem** [Guillopé-Zworski 1995, 1997] For $X = \Gamma \backslash \mathbb{H}^2$, let $\mathcal{R}_X = \{\text{Resonances}\}$. Then we have as $t \to +\infty$,

$$\#\{s \in \mathcal{R}_X \ : \ |s - 1/2| < t\} \asymp t^2.$$

Higher dimensions: Upper bound *[Patterson-Perry 2001, Cuevas-Vodev 2003, B 2008, Borthwick-Guillarmou 2016]*. Lower bound *[Perry 2003, Borthwick 2008]*.

**Theorem** [Guillopé-Lin-Zworski 2006] For $\sigma \leq \delta$, set

$$n(\sigma, T) = \#\{s \in \mathcal{R}_X \ : \ \Re(s) \geq \sigma \text{ and } |\Im(s) - T| \leq 1\}.$$

Then for all $\sigma$, as $T \to +\infty$, we have

$$n(\sigma, T) = O_\sigma(T^\delta).$$

Extensions to all convex co-compact hyperbolic manifolds by *[Dyatlov-Datchev 2013]*.

# *Spectral gaps*

**Theorem** ([Naud 2005, Bourgain-Dyatlov 2017])
There exists $\epsilon(\Gamma) > 0$ such that $\mathcal{R}_X \cap \{\delta - \epsilon \leq \Re(s) \leq \delta\} = \{\delta\}$.
Moreover, there exists $\epsilon_0(\delta) > 0$ such that
$\mathcal{R}_X \cap \{\delta - \epsilon_0 \leq \Re(s) \leq \delta\}$ is finite.
When $\delta > 1/2$, this statement follows readily from the discreteness of the spectrum below $1/4$.

**Theorem** [Bourgain-Dyatlov 2016]
There exists $\epsilon_1 > 0$ such that $\mathcal{R}_X \cap \{1/2 - \epsilon_1 \leq \Re(s)\}$ is finite.
**Conjecture** [JN12](*Essential Spectral Gap*)
Set $GAP(\Gamma) = \inf\{\sigma \in \mathbb{R} \ : \ \mathcal{R}_X \cap \{\sigma \leq \Re(s)\} \text{ is finite}\}$. Then $GAP(\Gamma) = \delta/2$.
If $\Gamma$ is cofinite, then we know (Selberg) that indeed $GAP(\Gamma) = \delta/2 = 1/2$. All previous results support the conjecture (or at least are consistent with it).

**Lower bounds:** Guillopé, Zworski: $\forall \epsilon > 0, \ \exists C_\epsilon > 0$, such that

$$N_{C_\epsilon}(T) = \Omega(T^{1-\epsilon}).$$

The proof uses a wave trace formula for resonances on $X$ and takes into account contributions from a *single* closed geodesic on $X$.

**Question:** Can one improve lower bounds taking into account contributions from *many* closed geodesics on $X$?

**Answer:** Yes, [JN10].

Guillopé, Lin, Zworski: let

$$\mathcal{D}(z) = \{\lambda \in \mathcal{R}_X : |\lambda - z| \leq 1\}$$

Then for all $z : \Im(z) \leq C$, we have $\mathcal{D}(z) = O(|\Re(z)|^{\delta})$.
Let $A > 0$, and let $W_A$ denote the logarithmic neighborhood of the real axis:

$$W_A = \{\lambda \in \mathbb{C} : \Im\lambda \leq A \log(1 + |\Re\lambda|)\}$$

**Theorem 5.** Let $X$ be a geometrically finite hyperbolic surface of infinite area, and let $\delta > 1/2$. Then there exists a sequence $\{z_i\} \in W_A, \Re(z_i) \to \infty$ such that

$$\mathcal{D}(z_i) \geq (\log|\Re(z_i)|)^{\frac{\delta - 1/2}{\delta} - \epsilon}.$$

**Corollary:** If $\delta > 1/2$, then $W_A \cap \mathcal{R}_X$ is different from a lattice. Examples of $\Gamma$ such that $\delta(\Gamma) > 1/2$ are easy to construct. Pignataro, Sullivan: fix the topology of $X$. Denote by $l(X)$ the maximum length of the closed geodesics that form the boundary of $N$. Then $\lambda_0(X) \leq C(X)l(X)$, where $C = C(X)$ depends only on the topology of $X$. By Patterson-Sullivan, $\lambda_0 < 1/4 \Leftrightarrow \delta > 1/2$, so letting $l(X) \to 0$ gives many examples. Proof of Theorem 5 uses (a version of Selberg) trace formula due to Guillopé and Zworski, and Dirichlet box principle.

Theorem 5 gives a *logarithmic* lower bound $\mathcal{D}(z_i) \geq (\log|\Re(z_i)|)^{\frac{\delta-1/2}{\delta}-\epsilon}$ for an infinite sequence of disks $D(z_i, 1)$. Conjecture of Guillopé and Zworski would imply that $\forall \epsilon > 0\ \exists \{z_i\}$ such that $\mathcal{D}(z_i) \geq |\Re(z_i)|^{\delta-\epsilon}$.

**Question:** can one get *polynomial* lower bounds for some particular groups $\Gamma$?

**Answer:** Yes. **Idea:** look at infinite index subgroups of arithmetic groups, and use methods of Selberg-Hejhal.

**Theorem 6.** Let $\Gamma$ be an infinite index geom. finite subgroup of an arithmetic group $\Gamma_0$ derived from a quaternion algebra. Let $\delta(\Gamma) > 3/4$. Then $\forall \epsilon > 0, \forall A > 0$, there exists $\{z_i\} \subset W_A, \Re(z_i) \to \infty$, such that

$$\mathcal{D}(z_i)) \geq |\Re(z_i)|^{2\delta-3/2-\epsilon}.$$

**Key ideas: arithmetic case.**
Number of closed geodesics on $X$:

$$\#\{\gamma : l(\gamma) < T\} \sim \frac{e^{\delta T}}{\delta T}, \qquad T \to \infty.$$

Number of *distinct* closed geodesics in the arithmetic case: for $\Gamma$ derived from a quaternion algebra, one has

$$\#\{L < T : L = l(\gamma)\} \ll e^{T/2}.$$

Accordingly, for $\delta > 1/2$, there exists *exponentially large* multiplicities in the length spectrum.
Distinct lengths are well-separated in the arithmetic case: for $l_1 \neq l_2$, we have

$$|l_1 - l_2| \gg e^{-\max(l_1, l_2)/2}.$$

Ex: $M_1, M_2 \in \mathrm{SL}(2, \mathbb{Z}), \mathrm{tr} M_1 \neq \mathrm{tr} M_2$ then
$|\mathrm{tr} M_1 - \mathrm{tr} M_2| = 2|\cosh(l_1/2) - \cosh(l_2/2)| \geq 1.$

We also use a trace formula (Guillopé, Zworski): Let $\psi \in C_0^\infty((0, +\infty))$, and $N$ - Nielsen region. Then (in case there are no cusps)

$$\sum_{\lambda \in \mathcal{R}_X} \widehat{\psi}(\lambda) = -\frac{V(N)}{4\pi} \int_0^{+\infty} \frac{\cosh(t/2)}{\sin^2(t/2)} \psi(t) dt$$
$$+ \sum_{\gamma \in \mathcal{P}} \sum_{k \geq 1} \frac{l(\gamma)\psi(kl(\gamma))}{2\sinh(kl(\gamma)/2)},$$

where $\mathcal{P} = \{\textit{primitive closed geodesics on } X\}$.

It follows from a recent result of Lewis Bowen that in every co-finite or co-compact arithmetic Fuchsian group, one can find infinite index convex co-compact subgroups with $\delta$ arbitrarily close to 1 (and in particular $> 3/4$). A. Gamburd considered infinite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and constructed subgroups $\Lambda_N$ such that $\delta(\Lambda_N) \to 1$ as $N \to \infty$. It was shown in [JN10] that subgroups $\Gamma_N$ of $\Lambda_N$ (of index two) provide examples of "arithmetic" groups with $\delta(\Gamma_N) > 3/4$ for large enough $N$. Related questions were also considered by Bourgain and Kantorovich.
The results of [JN10] can probably be generalized to hyperbolic 3-manifolds.

**Existence of sharp resonances**

Sharp resonances are non-trivial resonances (other than $\delta$) that are the closest to $\Re(s) = \min\{1/2, \delta\}$.

**Theorem** [Guillopé-Zworski 1999] For $\sigma \leq \delta$, set

$$N(\sigma, T) = \#\{s \in \mathcal{R}_X \;:\; \Re(s) \geq \sigma \text{ and } |\Im(s)| \leq T\}.$$

Then for all $\epsilon > 0$, one can find $\sigma_\epsilon$ such that

$$N(\sigma_\epsilon, T) = \Omega(T^{1-\epsilon}).$$

**Theorem** ([JN12])

For all $\varepsilon > 0$, $\mathcal{R}_X \cap \{\delta/2 - \delta^2 - \varepsilon \leq \Re(s)\}$ is *infinite*. If in addition $\Gamma$ is arithmetic with $\delta > 1/2$ then there are infinitely many resonances in

$$\{\Re(s) \geq \delta/2 - 1/4 - \epsilon\}.$$

## Large Galois covers

Let $\Gamma$ be a fixed convex co-compact Fuchsian group. Any normal subgroup $\Gamma_j \subset \Gamma$ with finite index yields a Galois cover

$$\Gamma_j \backslash \mathbb{H}^2 \to \Gamma \backslash \mathbb{H}^2$$

with Galois group $G_j = \Gamma / \Gamma_j$. Limit sets of $\Gamma_j$ are the same: $\Lambda(\Gamma_j) = \Lambda(\Gamma)$.

**Theorem 7** [Jakobson-Naud 2016] Assume that we have a family $\Gamma_j$ as above with $|G_j| \to \infty$. Let $\mathbf{Res}_j$ denote resonances of $X_j := \Gamma_j \backslash \mathbb{H}^2$, then for all $\epsilon, \widetilde{\epsilon} > 0$, as $j \to \infty$ we have

$$C_\epsilon^{-1} |G_j| \leq \# \mathbf{Res}_j \cap \{|s| \leq \log^{1+\epsilon}(\log |G_j|)\} \leq |G_j|^{1+\widetilde{\epsilon}}.$$

Examples: congruence subgroups given by:

$$\Gamma \subset SL_2(\mathbb{Z}), \ \Gamma_p = \{\gamma \in \Gamma \ : \ \gamma \equiv Id \mod p\},$$

but also *abelian covers* where $G_j$ is a sequence of finite Abelian groups (ex: cyclic covers).

Related work for covers of compact manifolds:

H. Huber, *On the spectrum of the Laplace operator on compact Riemann surfaces*, Comm. Math. Helv. 57 (1982), 627-647.

R. Brooks, *The spectral geometry of a tower of coverings.* JDG 23:97-107, 1986. D. L. de George and N. R. Wallach. *Limit formulas for multiplicities in* $L^2(\Gamma\backslash G)$. Ann. of Math. 107:133-150, 1978.

H. Donnelly. *On the spectrum of towers*. Proc. Amer. Math. Soc. 87:322-329, 1983.

E. Le Masson and T. Sahlsten. *Quantum ergodicity and Benjamini-Schramm convergence of hyperbolic surfaces.* arXiv:1605.05720v3

Abert, Bergeron et al: *On the growth of* $L^2$ *-invariants for sequences of lattices in Lie groups.* Ann. Math. 185:711-790, 2017.

These results say (in the simplest form) that if $X_j$ is a sequence of compact hyperbolic surfaces with $\mathrm{Inj}(X_j) \to \infty$, then for any compact interval $I \subset (1/4, +\infty)$,

$$\#\mathbf{Sp}_j \cap I \sim C_I \mathrm{Vol}(X_j),$$

where $C_I$ is some "explicit" constant depending only on $I$.

**Abelian Covers** Here we assume that $\Gamma_j \lhd \Gamma$ is given by

$$\Gamma_j = Ker(\varphi_j),$$

where $\varphi_j : \Gamma \to G_j$ is an onto morphism, with abelian image $G_j$. The corresponding surface is denoted by

$$X_j := \Gamma_j \backslash \mathbb{H}^2.$$

*Basic example* (cyclic covers). We have the identity

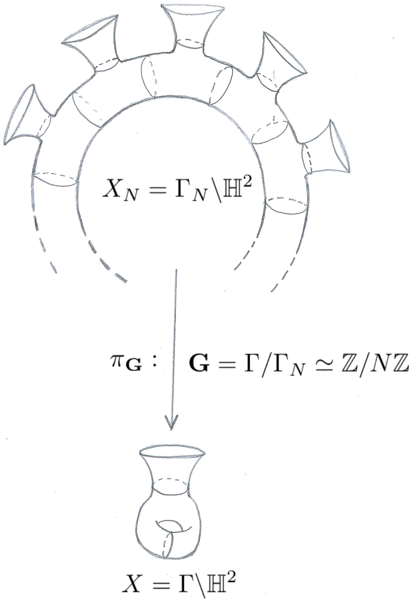
$$\Gamma^{ab} \simeq H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^r,$$

for some $r \geq 2$. Then $\Pi_N : \mathbb{Z}^r \to \mathbb{Z}/N\mathbb{Z}$ given by

$$\Pi_N(x_1, \ldots, x_r) = x_1 \mod N,$$

produces a normal subgroup $\Gamma_N \lhd \Gamma$ such that the covering group is $\mathbb{Z}/N\mathbb{Z}$.

# *A picturesque view of cyclic covers*



In the abelian case, sharp resonances/low eigenvalues equidistribute near $\delta$.

**Theorem 8** [JNS17] Assume that $\Gamma$ is non elementary. Assume that $G_j$ is Abelian, and $|G_j| \to \infty$. Then up to sequence extraction, there exists a neighbourhood $\mathcal{U} \ni \delta$ such that for all $j$ large, $\textbf{Res}(X_j) \cap \mathcal{U} \subset \mathbb{R}$ and for all $\varphi \in C_0(\mathcal{U})$,

$$\lim_{j \to \infty} \frac{1}{|G_j|} \sum_{\lambda \in \textbf{Res}(X_j)} \varphi(\lambda) = \int \varphi d\mu,$$

where $\mu$ is an absolutely continuous measure supported on an interval $[a, \delta]$, for some $a < \delta$.

In the *compact* and finite area case, Randol (1974) and Selberg showed that by moving to a finite abelian cover, one can have has many low eigenvalues in $[0, 1/4]$ as wanted.

Theorem 8 is a quantitative version of that, which works for resonances!

A related result:
**Theorem** $8\frac{3}{4}$**:** Let $X = \Gamma\backslash\mathbb{H}^2$ have at least one cusp. Assume that $G_j$ is Abelian, and $|G_j| \to \infty$. Then for all $\epsilon > 0$, there exists $j > 0$ s.t. $X_j = \Gamma_j\backslash\mathbb{H}^2$ has at least one nontrivial resonance $s$ with $|\delta - s| \leq \epsilon$.

## Congruence covers

Here we have

$$\Gamma \subset SL_2(\mathbb{Z}),\ \Gamma_p = \{\gamma \in \Gamma\ :\ \gamma \equiv Id \bmod p\},$$

and for all $p$ large enough prime number, we have

$$\Gamma/\Gamma(p) \simeq SL_2(\mathbb{F}_p).$$

Low eigenvalues and sharp resonances exhibit some rigidity when $|G_j| \to \infty$.

[Gamburd, 2002]. If $\Gamma \subset SL_2(\mathbb{Z})$ is a finitely generated subgroup and $\delta > 5/6$ then for all $p$ large,

$$\mathbf{Res}(\Gamma_p \backslash \mathbb{H}^2) \cap \{\Re(s) > 5/6\} = \mathbf{Res}(\Gamma \backslash \mathbb{H}^2) \cap \{\Re(s) > 5/6\}.$$

[Oh-Winter, 2016]. If $\Gamma \subset SL_2(\mathbb{Z})$ is a finitely generated subgroup, then there exists $\epsilon(\Gamma)$ such that for all $p$ large

$$\mathbf{Res}(\Gamma_p \backslash \mathbb{H}^2) \cap \{\Re(s) > \delta - \epsilon\} = \mathbf{Res}(\Gamma \backslash \mathbb{H}^2) \cap \{\Re(s) > \delta - \epsilon\}$$

$$= \{\delta\}.$$

All of these results can be seen as part of a Generalized Selberg's conjecture.

**Conjecture** For all finitely generated subgroups $\Gamma$ of $SL_2(\mathbb{Z})$, for all $\epsilon > 0$ we have for all $p$ large,

$$\mathbf{Res}(\Gamma_p \backslash \mathbb{H}^2) \cap \{\Re(s) > \delta/2 + \epsilon\} = \mathbf{Res}(\Gamma \backslash \mathbb{H}^2) \cap \{\Re(s) > \delta/2 + \epsilon\}.$$

**Theorem 9** [Jakobson-Naud 2016] There exists a computable $\alpha(\Gamma) > 0$ such that for all $\sigma \geq \delta/2$, we have

$$\#\mathbf{Res}_j \cap \{\Re(s) \geq \sigma \text{ and } |\Im(s)| \leq T\} \leq C_T |G_p|^{1+(\delta-2\sigma)\alpha(\Gamma)},$$

here $G_p = \Gamma/\Gamma_p \simeq SL_2(\mathbb{F}_p)$, and $|G_p| \asymp p^3$.

**Theorem 10** [Jakobson-Naud 2017] Let $\Gamma \subset SL_2(\mathbb{Z})$ be a convex co-compact subgroup and $\delta > 3/4$. Let $\mathbf{Res}_p$ denote the resonances of $X_p := \Gamma_p \backslash \mathbb{H}^2$. We set

$$N_\epsilon(p) := \#\mathbf{Res}_p \cap \left\{ |\Im(s)| \leq \log^{1+\epsilon}(\log p) \text{ and } \Re(s) > \delta - \frac{3}{4} - \epsilon \right\}.$$

Then for all $\epsilon > 0$, for all $\widetilde{\epsilon} > 0$, as $p \to \infty$ we have for large $p$.

$$\frac{p-1}{2} \leq N_\epsilon(p) \leq p^{3+\widetilde{\epsilon}}.$$

Remark:

$$0-\mathrm{Vol}(X_p) \asymp \chi(X_p) \asymp |G_p| \asymp p^3$$

*Extensions of Theorem 10 to more general "arithmetic covers" (arising from quaternion algebras) are doable*.