Numerical ranges of restricted shifts, norms of TTOs, and the role of Banach algebras

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Conference in honor of Tom Ransford 2018

The main ingredients

1. The numerical range

A an $n \times n$ matrix.

The *numerical range* of A is $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$.

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$$W(A_1) = \{0\}, W(A_2) = \{z : |z| \le 1/2\}.$$

Bedlewo, 2009

Crouzeix's conjecture

Michel Crouzeix 2006: "Open problems on the numerical range and functional calculus"

Conjecture (2004): For any polynomial $p \in \mathbb{C}[z]$ and A an $n \times n$ matrix the inequality holds:

$$\|p(A)\| \leq C \max |p(z)|_{z \in W(A)}.$$

The best constant should be $C = 2$.

Let $p(z) = z$ and $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$LHS = 1 \text{ and } RHS = C \cdot 1/2.$$

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4. (D. Choi) 3×3 matrices that are “nearly” Jordan blocks.
5. (Crouzeix, Palencia) Best constant is between 2 and $1 + \sqrt{2}$.

$A(\Omega)$ continuous functions on $\overline{\Omega}$ holomorphic on Ω.

Lemma

Let T be a bounded operator and Ω be a bounded open set containing the spectrum of T. Suppose that for each $f \in A(\Omega)$ there exists $g \in A(\Omega)$ such that

$$\|g\|_\Omega \leq \|f\|_\Omega \text{ and } \|f(T) + g(T)^\star\| \leq 2\|f\|_\Omega.$$

Then

$$\|f(T)\| \leq (1 + \sqrt{2})\|f\|_\Omega, f \in A(\Omega).$$

Ransford and Schwenninger gave a short proof of this lemma and show that in this lemma, the constant $(1 + \sqrt{2})$ is sharp.

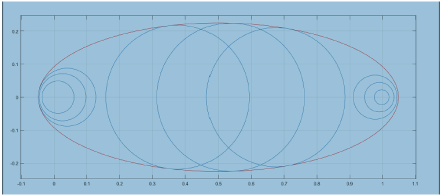
Numerical range basics

Theorem (Elliptical range theorem.)

The numerical range of a 2×2 matrix A is an elliptical disk with foci at the eigenvalues a and b of A and minor axis of length

$$(tr(A^\star A) - |a|^2 - |b|^2)^{1/2}.$$

Use computations or envelopes.



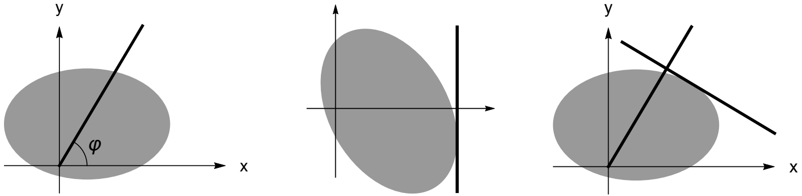
Theorem (Toeplitz Hausdorff theorem.)

The numerical range of a matrix is convex.

2. Projective Geometry

Kippenhahn: Finding the numerical range

Idea: Find the maximum eigenvalue of $(A + A^\star)/2$. Then rotate A and repeat.

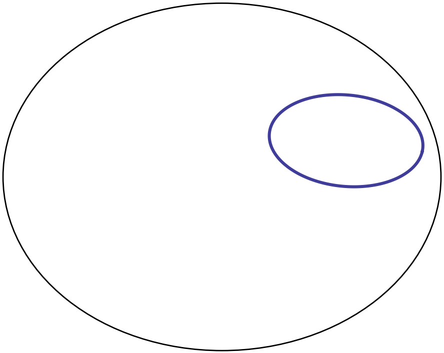


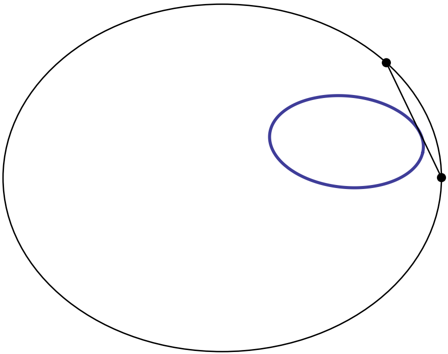
Theory of envelopes and projective geometry

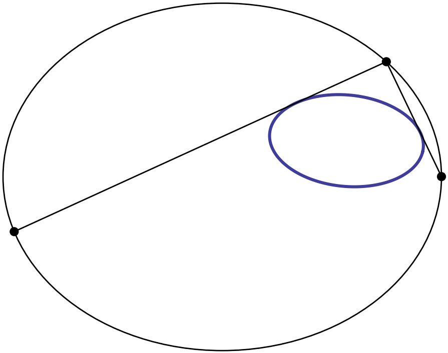
Poncelet's theorem, 1813

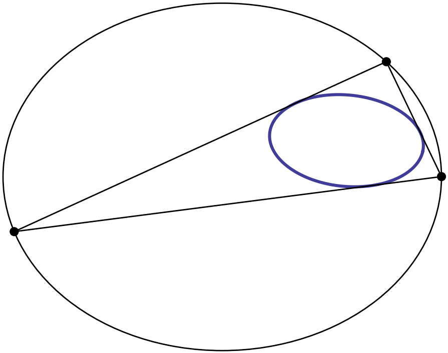
Theorem

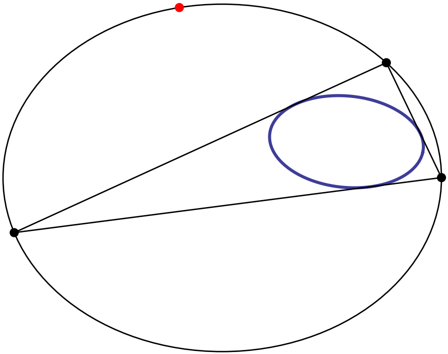
(Poncelet's Porism, ellipse version) Given one ellipse inside another, if there exists one circuminscribed (simultaneously inscribed in the outer and circumscribed on the inner) n -gon, then any point on the boundary of the outer ellipse is the vertex of some circuminscribed n-gon.

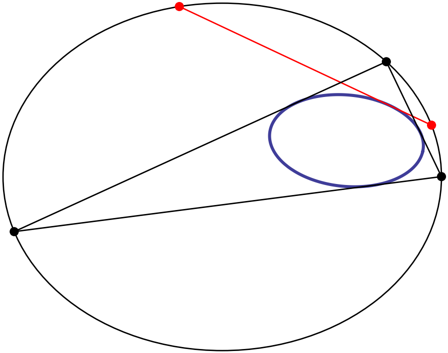


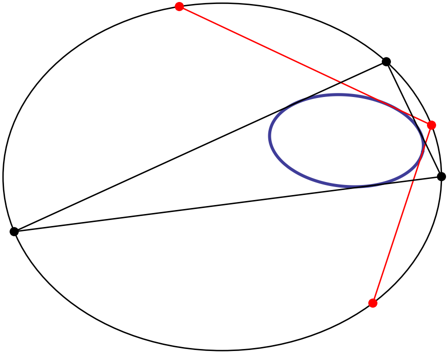


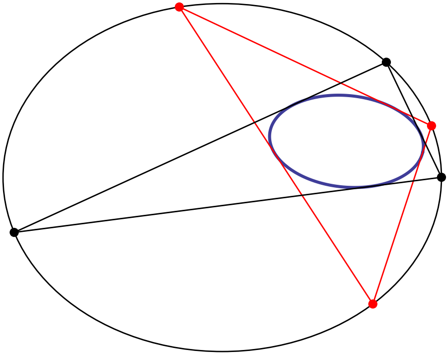








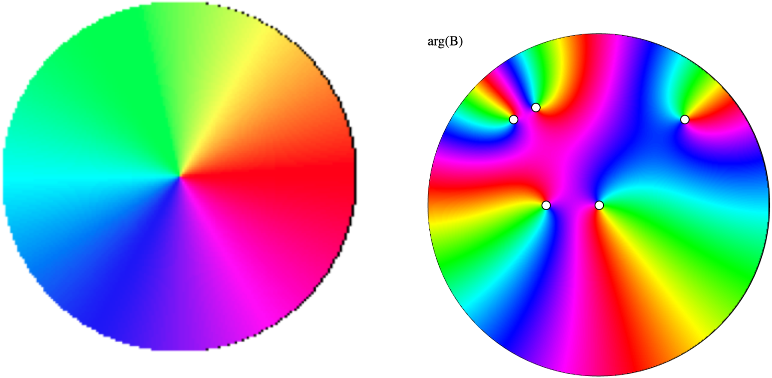




In honor of the 200th anniversary of Poncelet's theorem, Halbeisen and Hungerbühler gave a beautiful and accessible proof of the theorem.

3. Operator theory and Blaschke products

$$B(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}, \text{ where } a_j \in \mathbb{D}, |\lambda| = 1.$$



Visualizing Blaschke products

Compressions of the shift with finite Blaschke symbol

H^2 is the Hardy space; $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

An inner function I is a bounded analytic function on $\mathbb{D}$ with radial limits $|I^\star| = 1$ a.e.

S is the shift operator $S : H^2 \to H^2$ defined by $[S(f)](z) = zf(z)$;

The adjoint is $[S^\star(f)](z) = (f(z) - f(0))/z$.

Theorem (Beurling's theorem)

The nontrivial invariant subspaces under S are $UH^2 = \{Uh : h \in H^2\}$, where U is an inner function.

Subspaces invariant under the adjoint, $S^\star$ are $K_U := H^2 \ominus UH^2$.

What's the matrix representing S_B, B Blaschke?

$$B(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}, \text{ where } a_j \in \mathbb{D}, |\lambda| = 1.$$

$K_B := H^2 \ominus BH^2$.

Consider K_B where $B(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-\overline{a_j}z}$.

Consider the Szegö kernel: $g_a(z) = \dfrac{1}{1 - \overline{a}z}$.

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If a_j are distinct, $K_B = \text{span}\{g_{a_j} : j = 1, \ldots, n\}$.

Consider the compression of the shift: $S_B : K_B \to K_B$ defined by

$$S_B(f) = P_B S(f),$$

where P_B is the orthogonal projection from H^2 onto K_B.

Applying Gram-Schmidt to the kernels we get the Takenaka-Malmquist basis: Let $b_a(z) = \frac{z-a}{1-\overline{a}z}$ and

$$\{\frac{\sqrt{1-|a_1|^2}}{1-\overline{a_1}z}, b_{a_1}\frac{\sqrt{1-|a_2|^2}}{1-\overline{a_2}z}, \ldots \prod_{j=1}^{k-1} b_{a_j}\frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}z}, \ldots\}.$$

Matrix for S_B with B finite, $a_1, \ldots, a_n$ the zeros of B

The $n \times n$ matrix A is

$$\begin{bmatrix} a_1 & \sqrt{1-|a_1|^2}\sqrt{1-|a_2|^2} & \ldots & (\prod_{k=2}^{n-1}(-\overline{a_k}))\sqrt{1-|a_1|^2}\sqrt{1-|a_n|^2} \\ 0 & a_2 & \ldots & (\prod_{k=3}^{n-1}(-\overline{a_k}))\sqrt{1-|a_2|^2}\sqrt{1-|a_n|^2} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & a_n \end{bmatrix}$$

What's the connection?

Unitary dilations

$$U_\lambda = \begin{bmatrix} A & \text{stuff}(\lambda) \\ \text{stuff}(\lambda) & \text{stuff}(\lambda) \end{bmatrix}$$

1. The eigenvalues of U_λ are the values $zB(z)$ maps to λ;
2. $W(U_\lambda)$ is the polygon formed with the points $zB(z)$ identifies.
3. $W(A) \subseteq \bigcap\{W(U_\lambda) : \lambda \in \mathbb{D}\}$. We'll see why this is true in a moment. First let's talk about why we care about this.

Enter Halmos

What do unitary dilations of an operator T know about T?

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Specifically, is $W(A) = \bigcap\{W(U_\lambda) : U \text{ a unitary dilation of } A\}$

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$W(A) \subseteq \bigcap\{W(U_\lambda) : \lambda \in \mathbb{D}\}$.

Fact: Because $\text{rank}(I - S_B^\star S_B) = \text{rank}(I - S_B S_B^\star) = 1$, these operators have unitary 1-dilations. So we've added one row and one column.

Let $V = [I_n, 0]$ be $n \times (n+1)$. Then $A = VU_\lambda V^t$ and $V^t x = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\|V^t x\| = 1$.

$$\langle Ax, x\rangle = \langle VU_\lambda V^t x, x\rangle = \langle U_\lambda V^t x, V^t x\rangle.$$

$\|V^t x\| = 1$ implies the containment.

For each $\lambda \in \mathbb{T}$, we get a unitary 1-dilation of S_B:

$$b_{ij} = \begin{cases} a_{ij} & \text{if } 1 \leq i,j \leq n, \\ \lambda\left(\prod_{k=1}^{j-1}(-\overline{a_k})\right)\sqrt{1-|a_j|^2} & \text{if } i = n+1 \text{ and } 1 \leq j \leq n, \\ \left(\prod_{k=i+1}^{n}(-\overline{a_k})\right)\sqrt{1-|a_i|^2} & \text{if } j = n+1 \text{ and } 1 \leq i \leq n, \\ \lambda\prod_{k=1}^{n}(-\overline{a_k}) & \text{if } i = j = n+1. \end{cases}$$

The eigenvalues of U_λ are the values for which $zB(z) = \lambda$, and that's all we need to find $W(U_\lambda)$ (=convex hull of the eigenvalues).

These unitary dilations know everything

The class $\mathcal{S}_n$: Matrices with no eigenvalues of modulus 1, contractions (completely non-unitary contractions) with $\text{rank}(I - T^\star T) = 1$.

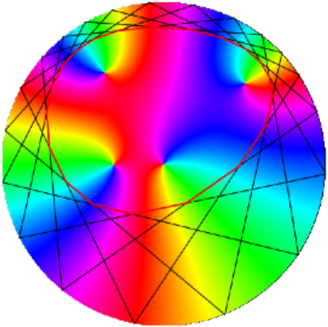
Theorem (Gau, Wu)

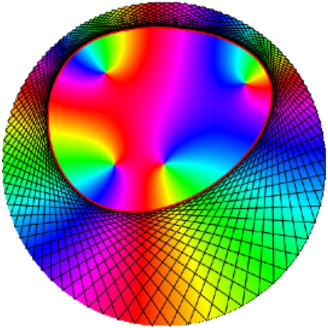
For $T \in \mathcal{S}_n$ and any point $\lambda \in \mathbb{T}$ there is an $(n+1)$-gon inscribed in $\mathbb{T}$ that circumscribes the boundary of $W(T)$ and has λ as a vertex.

The points where $zB(z) = \lambda$ are the vertices of a polygon. When we intersect the convex hull of these vertices we get $W(S_B)$:

$$W(S_B) = \bigcap \{W(U) : U \text{ unitary dilation of } S_B\}.$$

Connecting points identified by $zB(z)$/looking at $W(U_\lambda)$





Upshot

It's hard to say something about the shape of the numerical range for high degree...as in degree ≥ 3. So we ask an easier question.

The numerical radius of T is

$$r(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

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$$r(T) \leq \|T\| \leq 2r(T).$$

When $r(T) \leq 1$ we have a local version:

Theorem (Klaja, Mashreghi, Ransford)

If T is an operator on a Hilbert space $\mathcal{H}$ with $r(T) \leq 1$, then

$$\|Tx\|^2 \leq 2 + 2\sqrt{1 - |\langle Tx, x\rangle|^2} \qquad (x \in \mathcal{H}, \|x\| \leq 1).$$

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(Gaaya, 2012) Studied the numerical radius of S_B when

$$B(z) = \left(\frac{z - \alpha}{1 - \overline{\alpha} z} \right)^n .$$

Showed $W(S_B)$ is symmetric; computed bounds on $r(S_B)$.

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The Banach algebra approach

Basic connection

(Lumer, 1961)

$$\max\{\Re\lambda : \lambda \in W(T)\} = \lim_{a\to 0^+} (\|I + aT\| - 1)/a.$$

Replacing T by $e^{i\theta}T$ and maximizing yields the numerical radius.

Study $\|I + aS_B\|$ for B a finite Blaschke product, a small. It is known that $\|I + aS_B\| =$ dist$((1 + az)/B(z), H^\infty)$, so we have more techniques at our disposal.

Baby steps: $\|p(S_B)\|$

Foias-Tannenbaum, 1987

Let $|a| < 1$ and for $\rho > 0$. Let

$$P_\rho = I - \frac{1}{4\rho^2}(I + aS_B)(I + \overline{a}S_B^\star).$$

The largest ρ for which P_ρ is singular is $\|(I + aS_B)/2\|$.

We compute ρ and obtain $r(S_B)$.

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Theoretically...To say more, use

Theorem

Let B be a Blaschke product with real zeros. Then $r(S_B)$ is attained on the real line.

makes it easy to compute special cases, including Gaaya's result.

Sarason's work

Theorem

Let B be a Blaschke product with real zeros. Then $r(S_B)$ is attained on the real line.

- $\|I + aS_B\| \leq \gamma$ precisely when there exist $g, h \in H^\infty$ and $\|h\|_\infty \leq \gamma$ with

$$1 + az = B(z)g(z) + h(z).$$

- If we solve $f(z_k) = 1 + tz_k$ for $t > 0$ and $\|f\|_\infty \leq \gamma$, then we can solve $h(z_k) = 1 + te^{i\theta}z_k$ with $\|h\|_\infty \leq \gamma + o(t)$.

Examples

1. Blaschke products of degree 2: Can compute $r(S_B)$ and compare to the elliptical range theorem.

2. Blaschke products of degree 3: Let B have zeros $a, b, c \in \mathbb{R}$ and $\alpha = a + b + c$, $\beta = ab + ac + bc$, then

$$w(S_B) = \max \left\{ \left| \frac{\alpha + \sqrt{\alpha^2 - 8(\beta - 1)}}{4} \right|, \left| \frac{\alpha - \sqrt{\alpha^2 - 8(\beta - 1)}}{4} \right| \right\}.$$

3. (Gaaya) Obtain numerical radius of S_B for $B(z) = \left(\frac{\alpha - z}{1 - \overline{\alpha} z}\right)^n$.

This gives us the following possible approaches

- Numerical range and projective geometry;
- Operator theory
- Banach algebra techniques and distance estimates

Generalizing $S_B = P_{K_B} S$ on K_B

Let $a > 0$: $\|I + aS_B\| =$ dist$((1 + az)/B(z), H^\infty)$, the norm of a truncated Toeplitz operator (TTO).

For $g \in L^\infty$, θ inner, define the TTO

$$A_g^\theta = P_{K_\theta} M_g.$$

For $g \in H^\infty$,

$$\|A_g^\theta\| = \text{dist}(\overline{\theta} g, H^\infty) = \|H_{\overline{\theta} g}\|$$

with $H_f : H^2 \to \overline{H_0^2}$ the Hankel operator defined by $H_f u = P_{\overline{H_0^2}}(fu)$.

(Sarason, 1967 Generalized interpolation on H^∞)

Other distance estimates and the Sarason algebra

An important algebra: the Sarason algebra

$$H^\infty + C(\mathbb{T}) = \{f + g : f \in H^\infty, g \in C(\mathbb{T})\}.$$

Sarason showed that

1. $H^\infty + C$ is a subalgebra of L^∞.
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But also plays an important role in operator theory (compactness).

Division in $H^\infty + C$

Theorem (Axler, G.; Guillory, Izuchi, Sarason)

Let B be an interpolating Blaschke product with zeros (z_n). If $f \in H^\infty + C$ and $f(z_n) \to 0$, then $\overline{b}f \in H^\infty + C$.

Remark: It's a sort of Schwarz lemma: If b is a finite product of interpolating Blaschke products and $g \in H^\infty + C$ is such that every zero of B is a zero of g of at least as high a multiplicity, then

$$|g| \leq |b| \|g\|_\infty$$

on $M(H^\infty) \setminus \mathbb{D}$.

An example in this setting

Theorem (Bessonov)

Let u be an inner function and $\varphi \in H^\infty + C(\mathbb{T})$. Then the truncated Toeplitz operator $A_\varphi^u : K_u \to K_u$ is compact if and only if $\varphi \in u(H^\infty + C(\mathbb{T}))$.

$$\|\varphi \overline{u} + H^\infty + C(\mathbb{T})\| = 0.$$

or u divides φ in $H^\infty + C(\mathbb{T})$

Before we looked at $\|(I + az)\overline{B} + H^\infty\|$, B finite. Now we generalize this.

Interpolating Blaschke products

A Blaschke product B is *interpolating* if the zero sequence (z_n) of B is an interpolating sequence for H^∞; i.e., given a bounded sequence (w_n), there exists $f \in H^\infty$ with $f(z_n) = w_n$ for all n.

Carleson: this is equivalent to the existence of $\delta > 0$ with

$$\inf_n \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - z_m \overline{z_n}} \right| \geq \delta.$$

Let

$$\delta_n := \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - z_m \overline{z_n}} \right|.$$

If $\delta_n \to 1$ as $n \to \infty$, the interpolating sequence is said to be a *thin interpolating sequence*.

A radial sequence (z_n) for which

$$(1 - |z_{n+1}|)/(1 - |z_n|) \to 0$$

as $n \to \infty$ is thin.

There are a lot of these sequences!

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There are a lot of these sequences! They have very nice properties!

1. Every sequence clustering on the unit circle contains a thin subsequence;
2. (Chalendar, Fricain, Timotin) Every Blaschke sequence can be rotated into a thin sequence.

Recall Bessov's condition for compactness:

$$\|\varphi\overline{u} + H^\infty + C(\mathbb{T})\| = [\|\varphi + u\,(H^\infty + C(\mathbb{T}))\,\| = 0.$$

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Theorem (G., Partington)

Let B be a thin interpolating Blaschke product with zero sequence (z_n) and let $f \in H^\infty + C(\mathbb{T})$. Then

$$\|f + B(H^\infty + C(\mathbb{T}))\| = \limsup |f(z_n)|.$$

This also gives you an estimate on the distance when B is an interpolating Blaschke product.

Bringing it back home

supp u is the closure of the union of the zeros of u and the support of the singular measure (if there is one).

Theorem (Ahern and Clark)

Let f be continuous and let u be an inner function. The operator A_f^u is compact if and only if $f(e^{i\theta}) = 0$ for all $e^{i\theta} \in$ supp $u \cap \mathbb{T}$.

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Theorem (G., Partington)

Let $f \in H^\infty + C(\mathbb{T})$ and let u be an inner function. The operator $A_f^{u^n}$ is compact for every $n \in \mathbb{N}$ if and only if

$$\lim_{|z| \to 1^-} |f(z)|(1 - |u(z)|) = 0.$$

Question

The symbol is not unique.

Chalendar, Fricain, and Timotin ask for an example of a compact TTO with symbol in $\theta(H^\infty + C(\mathbb{T}))$ that has no continuous symbol.

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Example. Let B be an ibp with zero sequence clustering at every point of $\mathbb{T}$ and let $f \in H^\infty + C$ with $f(z_n) \to 0$ but $f(z_n) \neq 0$ for all n. Then A_f^B is compact, but has no continuous symbol.

There's lots more to do here in all three areas:
Numerical range, distance estimates, operator norms

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