RKH spaces defined by Cesàro operators

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CESÀRO OPERATORS

Let $\alpha > 0$. For $x > 0$,

$$(\mathcal{C}_\alpha f)(x) := x^{-\alpha} \int_0^x (x-y)^{\alpha-1} f(y) \frac{dy}{\Gamma(\alpha)},$$

$$(\mathcal{C}_\alpha^* g)(x) := \int_x^\infty (y-x)^{\alpha-1} \frac{g(y)}{y^\alpha} \frac{dy}{\Gamma(\alpha)}$$

$$\mathcal{C}_\alpha \colon L_p(\mathbb{R}^+) \to L_p(\mathbb{R}^+), \quad 1 < p \leq \infty$$

$$\mathcal{C}_\alpha^* \colon L_p(\mathbb{R}^+) \to L_p(\mathbb{R}^+), \quad 1 \leq p < \infty.$$

Definition

$$\mathcal{T}_2^{(\alpha)}(t^\alpha) := \mathcal{C}_\alpha^*(L_2(\mathbb{R}^+))$$

$\mathcal{C}_\alpha^*$ injective, so:

$$\|f\|_{2,\alpha} := \|(\mathcal{C}_\alpha^*)^{-1} f\|_{L_2(\mathbb{R}^+)}, \quad f \in \mathcal{T}_2^{(\alpha)}(t^\alpha).$$

- Define

$$W^\alpha : \mathcal{T}_2^{(\alpha)}(t^\alpha) \xrightarrow{(\mathcal{C}_\alpha^*)^{-1}} L_2(\mathbb{R}^+) \xrightarrow{\mu_{x^{-\alpha}}} L_2(\mathbb{R}^+, x^{-\alpha})$$

  that is,

$$W^\alpha f(x) := x^{-\alpha}\left[(\mathcal{C}_\alpha^*)^{-1} f\right](x), \quad f \in \mathcal{T}_2^{(\alpha)}(t^\alpha), x > 0.$$

  Weyl fractional derivative, with inverse

$$W^{-\alpha} g(t) := \int_t^\infty (s-t)^{\alpha-1} g(s) \frac{ds}{\Gamma(\alpha)}$$

- Thus, $\mathcal{T}_2^{(\alpha)}(t^\alpha)$: f, there exist $W^\beta f$ $(0 \leq \beta < \alpha)$ and $W^\alpha F$ a. e. on $\mathbb{R}^+$,

$$\|f\|_{(\alpha),2} := \left(\int_0^\infty |W^\alpha f(x) x^\alpha|^2 dx\right)^{1/2} < \infty.$$

  For $\beta > \alpha$, $\quad \mathcal{T}_2^{(\beta)}(t^\beta) \subset \mathcal{T}_2^{(\alpha)}(t^\alpha) \subset \mathcal{T}_2^{(0)}(t^0) = L_2(\mathbb{R}^+)$.

  - $\mathcal{T}_1^{(n)}(t^n)$ (W. Arendt); $\alpha > 0$ (P. J. Miana).

POINT ESTIMATE

- $\mathcal{T}_2^{(\alpha)}(t^\alpha)$ is a RKH space if $\alpha > \frac{1}{2}$:

  $\forall f \in \mathcal{T}_2^{(\alpha)}(t^\alpha)$ and $x > 0$,

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (y-x)^{\alpha-1} W^\alpha f(y)dy$$

  and therefore

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{(y-x)^{\alpha-1}}{y^\alpha} |y^\alpha W^\alpha f(y)| dy \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^\infty \frac{(y-x)^{2\alpha-2}}{y^{2\alpha}} dy \right)^{1/2} \left(\int_0^\infty |y^\alpha W^\alpha f(y)|^2 \, dy \right)^{1/2} \\ &= \frac{1}{\Gamma(\alpha)\sqrt{2\alpha-1}} \frac{1}{\sqrt{x}} \|f\|_{(\alpha),2}. \end{aligned}$$

- If $\alpha \leq \frac{1}{2}$, $\mathcal{T}_2^{(\alpha)}(t^\alpha)$ is not a RKH space (later on).

REPRODUCING KERNEL

Reproducing kernel in $\mathcal{T}_2^{(\alpha)}(t^\alpha)$?

- $$f(x) = \int_0^\infty W^\alpha f(y) y^\alpha (y-x)_+^{\alpha-1} y^{-\alpha} \frac{dy}{\Gamma(\alpha)}, \; x > 0.$$
- Assume there exists RK $k_\alpha(\cdot,\cdot)$ for $\mathcal{T}_2^{(\alpha)}(t^\alpha)$:

$$f(x) = \langle f, k_\alpha(\cdot, x)\rangle_{(\alpha),2} = \int_0^\infty W^\alpha f(y) y^\alpha \overline{W^\alpha k_\alpha(y,x) y^\alpha} dy$$

- Then

$$\int_0^\infty \varphi(y) \left(\frac{(y-x)_+^{\alpha-1}}{y^\alpha \Gamma(\alpha)} - \overline{W^\alpha k_\alpha(y,x) y^\alpha} \right) dy = 0, \forall \varphi \in C_c^{(\infty)}(0,\infty);$$

- Hence,

$$\frac{(y-x)_+^{\alpha-1}}{y^\alpha \Gamma(\alpha)} = W^\alpha k_\alpha(y,x) y^\alpha \text{ in } L_2(\mathbb{R}^+) \iff \alpha > \frac{1}{2}$$

So,

$$\text{“}k_\alpha(y,x) = W^{-\alpha} \left[\frac{1}{\Gamma(\alpha)} \frac{(y-x)_+^{\alpha-1}}{y^{2\alpha}} \right]\text{”}$$

Let $\alpha > 1/2$.
Set

$$g_\alpha(t,r) := \frac{1}{\Gamma(\alpha)} \frac{(r-t)_+^{\alpha-1}}{r^\alpha}; \quad r,t>0.$$

Reproducing kernel of $\mathcal{T}_2^{(\alpha)}(t^\alpha)$:

$$k_\alpha(s,t) = \int_0^\infty g_\alpha(s,r) g_\alpha(t,r)\, dr; \quad s,t>0.$$

$$\|k_\alpha(\cdot,t)\|_{2,(\alpha)}^2 = \frac{1}{\Gamma(\alpha)^2(2\alpha-1)} \frac{1}{t}; \quad t>0.$$

$$k_\alpha(t,s) = \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \frac{1}{\text{máx}\{s,t\}}\, {}_2F_1\left(1-\alpha, 1, 1+\alpha, \frac{\text{mín}\{s,t\}}{\text{máx}\{s,t\}}\right).$$

$\alpha = n \in \mathbb{N} \Longrightarrow$ For $s,t>0$,

$$k_n(s,t) = \sum_{j=0}^{n} \frac{1}{(n+j)!\,(n-j-1)!} \frac{\text{mín}\{s,t\}^j}{\text{máx}\{s,t\}^{j+1}}$$

BROWNIAN MOTION

- $X_{\alpha,t}$:= Gaussian stochastic process s. t.

$$\mathrm{Cor}(X_{\alpha,s}, X_{\alpha,t}) = k_\alpha(s,t)$$

- B_t := Brownian motion, or Wiener process

  Integrated Brownian motion:

$$B_{1,t} := \int_0^t B_s \, ds \text{ and } B_{n,t} := \int_0^t B_{n-1,s} \, ds, \quad n \geq 2.$$

  Or

$$B_{n,t} = \int_0^t \frac{(t-s)^n}{n!} \, dB_s \quad \text{(stochastic integration)}$$

- For $\alpha > 0$,

$$B_{\alpha,t} \quad \text{fractional Brownian motion (fBM, for short)}$$

(B. B. Mandelbrot - J. W. Van Ness 1968)

Instead,

$$\mathcal{N}_{\alpha,t} := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dB_s \quad \text{[integrated WHITE NOISE]}$$

with

Covariance: $\nu_\alpha(s,t) = \int_0^{\text{mín}\{s,t\}} \frac{(s-u)^{\alpha-1}(t-u)^{\alpha-1}}{\Gamma(\alpha)^2}\, du$

Question.-

$$\text{Relation} \quad \mathcal{N}_{\alpha,t} \longleftrightarrow X_{\alpha,t} \ ?$$

- By change of variable,

$$\nu_\alpha(s,t) = (st)^{\alpha-1} k_\alpha\left(\frac{1}{s}, \frac{1}{t}\right) \quad \forall s,t > 0$$

- For ν_n, $n \in \mathbb{N}$, its RKHS is

$$\mathcal{B}_n = \{F \in H^{(n),2} : F^{(j)}(0) = 0 \ (0 \leq j \leq n-1)\}$$

Let $D^{-\alpha}\colon L_2(\mathbb{R}^+) \to L_2(\mathbb{R}^+)$ be given by

$$D^{-\alpha}G(x) := \int_0^x (x-y)^{-\alpha-1} G(y)\,\frac{dy}{\Gamma(\alpha)}, \quad G \in L_2(\mathbb{R}^+).$$

Definition

$$\mathfrak{B}_\alpha := D^{-\alpha}(L_2(\mathbb{R}^+)), \text{ with } \|F\|_{\mathfrak{B}} := \|(D^{-\alpha})^{-1}F\|_{L^2}$$

Set $F^{(\alpha)} = (D^{-\alpha})^{-1}F$. Thus,

$\forall F \in \mathfrak{B}_\alpha$ there exits a unique $F^{(\alpha)} \in L_2(\mathbb{R}^+)$ such that

$$F(x) = \int_0^x (x-y)^{\alpha-1} F^{(\alpha)}(y)\,\frac{dy}{\Gamma(\alpha)}; \quad x > 0.$$

- The mapping

$$J\colon f(x) \in L_2(\mathbb{R}^+) \mapsto \frac{1}{x}\, f\left(\frac{1}{x}\right) \in L_2(\mathbb{R}^+)$$

is an isometric isomorphism.

Lemma

Let $f \in \mathcal{T}_2^{(\alpha)}(t^\alpha)$. *Then*

$$F(x) = x^{\alpha-1} f(\frac{1}{x}) \Longleftrightarrow F^{(\alpha)}(x) = x^{-(\alpha+1)} W^\alpha f(\frac{1}{x})$$

Proof.

$$\begin{aligned}
F(x) &= x^{\alpha-1} f(\frac{1}{x}) \Longleftrightarrow F(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \int_{1/x}^{\infty} (t - \frac{1}{x})^{\alpha-1} W^\alpha f(t)\, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_{1/x}^{\infty} (x - \frac{1}{t})^{\alpha-1} t^{\alpha-1} W^\alpha f(t)\, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x - y)^{\alpha-1} y^{-(\alpha+1)} W^\alpha f(\frac{1}{y})\, dy = D^{-\alpha} \left(y^{-(\alpha+1)} W^\alpha f(\frac{1}{y}) \right).
\end{aligned}$$

SOBOLEV-BROWNIAN ISOMORPHISM

Proposition

The mapping $\Theta_\alpha\colon \mathcal{T}_2^{(\alpha)}(t^\alpha) \to \mathfrak{B}_\alpha$ *given by* $\Theta_\alpha\colon f \mapsto x^{\alpha-1} f(\frac{1}{x})$ *is an isometric isomorphism.*

Proof. $\Theta_\alpha = D^{-\alpha} \circ J \circ \mu_{x^\alpha} W^\alpha \colon \mathcal{T}_2^{(\alpha)}(t^\alpha) \to L_2(\mathbb{R}^+) \to L_2(\mathbb{R}^+) \to \mathfrak{B}_\alpha$ so that

$$\Theta_\alpha\colon f \mapsto x^\alpha W^\alpha f(x) \mapsto x^{-(\alpha+1)} W^\alpha f(\tfrac{1}{x}) \mapsto F \equiv x^{\alpha-1} f(\tfrac{1}{x}) \text{ with}$$

$$\int_0^\infty |x^{-(\alpha+1)} W^\alpha f(\frac{1}{x})|^2 dx = \int_0^\infty |t^\alpha W^\alpha f(t)|^2 dt.$$

Corollary

$\forall \alpha > 1/2$, $\mathfrak{B}_\alpha$ *is a RKHS with kernel*

$$\nu_\alpha(u,v) = \int_0^{\text{mín}\{u,v\}} (u-r)^{\alpha-1}(v-r)^{\alpha-1} \frac{dr}{\Gamma(\alpha)}.$$

Proof. $\nu_\alpha(u,v) = \int_0^\infty \Theta_\alpha(g(t-\cdot)(u)\Theta_\alpha(g(t-\cdot)(v)\frac{dt}{\Gamma(\alpha)};\ t = 1/r.$

$\mathfrak{B}_\alpha$ PROPERTIES

- $\forall \alpha, \beta > 0$,

$$\Theta_{\beta\alpha} \colon F \in L_2(x^{-\beta}) \mapsto x^{\alpha-\beta} F \in L_2(x^{-\alpha}) \qquad \text{isometric isomorphism}$$

- $\mathfrak{B}_\alpha \hookrightarrow L_2(x^{-\alpha})$: $F = \Theta_\alpha(f) \in \mathfrak{B}_\alpha$, $f \in \mathcal{T}_2^{(\alpha)}(t^\alpha)$,

$$\begin{aligned} \|F\|^2_{L_2(x^{-\alpha})} &= \int_0^\infty |x^{\alpha-1} f(\frac{1}{x})|^2 \frac{dx}{x^{2\alpha}} = \|f\|^2_{L_2(\mathbb{R}^+)} \\ &\le M_\alpha \|f\|_{(\alpha),2} \simeq \|F^{(\alpha)}\|_{L_2(\mathbb{R}_+)} = \|f\|^2_{\mathfrak{B}_\alpha} \end{aligned}$$

- Then, if $\beta > \alpha$,

$$\Theta_{\beta\alpha} \colon \mathfrak{B}_\beta \overset{\Theta_\beta}{\equiv} \mathcal{T}_2^{(\beta)}(t^\beta) \overset{\iota}{\hookrightarrow} \mathcal{T}_2^{(\alpha)}(t^\alpha) \overset{\Theta_\alpha}{\equiv} \mathfrak{B}_\alpha$$

$\mathcal{T}_2^{(\alpha)}(t^\alpha)$ "model" for fBM or White Noise ?

- Hardy space $H_2(\mathbb{C}^+)$: $\sup_{x>0} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(x+iy|^2 dy \right)^{1/2} < \infty$
- Paley-Wiener : $\mathcal{L}\colon L_2(\mathbb{R}^+) \to H_2(\mathbb{C}^+)$ isometry

$$f \mapsto \mathcal{L}f(z) := \int_0^\infty f(t)e^{-zt}dt, \quad z \in \mathbb{C}^+.$$

- Dzrbasjan (Sedleckii, $1 \le p \le \infty$)

Set $H_{2,\mathrm{rad}}(\mathbb{C}^+)$: F holomorphic on $\mathbb{C}^+$ s. t.

$$\sup_{-\pi/2<\theta<\pi/2} \left(\int_0^\infty |F(re^{i\theta})|^2 dr \right)^{1/2} < \infty.$$

Then

$$H_{2,\mathrm{rad}}(\mathbb{C}^+) \equiv H_2(\mathbb{C}^+)$$

isometrically.

- $\mathcal{T}_2^{(\alpha)}(t^\alpha) \hookrightarrow L_2(\mathbb{R}^+)$

Definition

Hardy-Sobolev space of order $\alpha > 0$:

$$H_2^{(\alpha)}(\mathbb{C}^+) := \mathcal{L}(\mathcal{T}_2^{(\alpha)}(t^\alpha)), \text{ with } \|\mathcal{L}f\|_{H_2^{(\alpha)}} := \|f\|_{(\alpha),2}.$$

- For $\alpha = 0$, $H_2(\mathbb{C}^+)$ is RKHS with $K(z,w) = (z+\overline{w})^{-1}$; $z, w \in \mathbb{C}^+$.

Proposition

For $\alpha > 1/2$, $H_2^{(\alpha)}(\mathbb{C}^+)$ is a RKHS with kernel

$$K_\alpha(z,w) = \int_0^\infty G_\alpha(z,r)\overline{G_\alpha(w,r)}\, dr, \quad z, w \in \mathbb{C}^+.$$

where

$$G_\alpha(z,r) := \mathcal{C}_\alpha(e^{-z\cdot}) = \frac{1}{r^\alpha}\int_0^r \frac{(r-u)^{\alpha-1}}{\Gamma(\alpha)} e^{-zu} du, z \in \mathbb{C}^+, r > 0.$$

Skectch of proof.-

$$\begin{aligned}(F|K_\alpha(\cdot,w))_{H_2^{(\alpha)}} &= F(w) = \mathcal{L}f(w) = \int_0^\infty \int_t^\infty W^\alpha f(s) \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} ds\ e^{-wt} dt \\ &= \int_0^\infty s^\alpha W^\alpha f(s) \frac{1}{s^\alpha} \int_0^s \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} e^{-wt} dt\ ds = (f|h_{\alpha,w})_{\mathcal{T}_2^{(\alpha)}}\end{aligned}$$

- Thus look for $K_\alpha(\cdot,w) = \mathcal{L}(h_{\alpha,w})$ such that

$$\begin{aligned}s^\alpha W^\alpha h_{\alpha,w}(s) &= s^{-\alpha} \int_0^s \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} e^{-\overline{w}t} dt \\ \iff & h_{\alpha,w}(s) = \int_s^\infty \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^r \frac{(r-t)^{\alpha-1}}{\Gamma(\alpha)} e^{-\overline{w}t} dt\ \frac{dr}{r^{2\alpha}}\end{aligned}$$

- Then

$$K_\alpha(z,w) = \int_0^\infty \left(\int_0^r \frac{(r-s)^{\alpha-1}}{r^\alpha} e^{-zs} \frac{ds}{\Gamma(\alpha)} \right) \left(\int_0^r \frac{(r-t)^{\alpha-1}}{r^\alpha} e^{-\overline{w}t} \frac{dt}{\Gamma(\alpha)} \right) dr.$$

Theorem

$$K_\alpha(z,w) = \int_0^1 \int_0^1 \frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)} \frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)} \frac{dx\ dy}{xz + y\overline{w}}$$

- If $\alpha = n \in \mathbb{N}$,

$$\begin{aligned} K_\alpha(z,w) &= \left[\sum_{j=0}^{n-1} \frac{1}{(n+j)!(n-1-j)!} \frac{\overline{w}^j}{z^{j+1}}\right] \log\left(\frac{z+\overline{w}}{\overline{w}}\right) \\ &+ \left[\sum_{j=0}^{n-1} \frac{1}{(n+j)!(n-1-j)!} \frac{z^j}{\overline{w}^{j+1}}\right] \log\left(\frac{z+\overline{w}}{z}\right) \\ &+ \sum_{j=1}^{n-1} \sum_{l=0}^{j-1} \frac{(-1)^{j+l}}{(j-l)(n+j)!(n-1-j)!} \frac{z^l}{\overline{w}^{l+1}} \\ &+ \sum_{j=1}^{n-1} \sum_{l=0}^{j-1} \frac{(-1)^{j+l}}{(j-l)(n+j)!(n-1-j)!} \frac{\overline{w}^l}{z^{l+1}}. \end{aligned}$$

DESCRIPTION of $H_2^{(\alpha)}(\mathbb{C}^+)$

- $H^2_{(n)}(\mathbb{C}^+)$: F holomorphic in $\mathbb{C}^+$ such that $z^n F^{(n)} \in H_2(\mathbb{C}^+)$ for all $n \in \mathbb{N}_0$;

$$\begin{aligned} F(z) &= \frac{(-1)^n}{\Gamma(n)} \int_{|z|e^{i\theta}}^{\infty\cdot e^{i\theta}} (\lambda - z)^{n-1} F^{(n)}(\lambda)\, d\lambda \\ &= \frac{(-1)^n}{\Gamma(n)} e^{i(n-1)\theta} \int_r^\infty (s-r)^{n-1} F^{(n)}(se^{i\theta})\, ds, \quad z = re^{i\theta}, |\theta| \le \pi/2. \end{aligned}$$

- Hardy's inequality $\Rightarrow F \in H_2(\mathbb{C}^+)$, $\quad \|F\|_{H_2} \le M_n \|z^n F^{(n)}\|_{H_2}$.

- *Complex fractional derivative*: $n > \alpha > 0$, $F \in H^2_{(n)}(\mathbb{C}^+)$,

$$W^\alpha F(z) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{|z|e^{i\theta}}^{\infty\cdot e^{i\theta}} (\lambda - z)^{n-\alpha-1} F^{(n)}(\lambda)\, d\lambda, \quad z = |z|e^{i\theta} \in \mathbb{C}^+$$

and then $\|z^\alpha W^\alpha F\|_2 \le \|z^n F^{(n)}\|_2$.

Definition of $H^2_{(\alpha)}(\mathbb{C}^+)$

Definition

$H^2_{(\alpha)}(\mathbb{C}^+)$:= completion of $H^2_{(n)}(\mathbb{C}^+)$ in

$$\|F\|_{H^2_{(\alpha)}} := \|z^\alpha W^\alpha F\|_{H_2}, \quad F \in H^2_{(n)}.$$

- Note: $\mathrm{ev}_z : H^2_{(\alpha)}(\mathbb{C}^+) \hookrightarrow H_2(\mathbb{C}^+) \xrightarrow{\mathrm{ev}_z} \mathbb{C}$ continuous, $\alpha > 1/2$.

- Laguerre functions; $m = 0, 1, \ldots$

$$\ell_m(x) := e^{-x/2} \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j}{j!} x^j, \quad x > 0.$$

Orthonormal basis in $L_2(\mathbb{R}_+)$. Hence,

$$\ell_{m,\alpha} := W^{-\alpha}(x^{-\alpha} \ell_m)$$

is O. B. of $\mathcal{T}_2^{(\alpha)}(t^\alpha)$.

SOME preliminary CALCULATION

So,

$$\mathcal{L}(\ell_{m,\alpha})(z) = \int_1^\infty \frac{2(u-1)^{\alpha-1}}{u^\alpha} \frac{(2z-u)^m}{(2z+u)^{m+1}} \frac{du}{\Gamma(\alpha)}$$

is OB in $H_2^{(\alpha)}(\mathbb{C}^+) = \mathcal{L}(\mathcal{T}_2^{(\alpha)}(t^\alpha))$.

• On the other hand,

$$B_{m,\alpha}(z) := \frac{(-1)^m}{2z}\mathcal{L}(\ell_{m,\alpha})(\frac{1}{4z}), \quad z \in \mathbb{C}^+, m = 0, 1, \ldots$$

is an OB for $H^2_{(\alpha)}(\mathbb{C}^+)$:

$$F \in H_2(\mathbb{C}^+) \mapsto W^{-\alpha}(z^{-\alpha}F) \in H^2_{(\alpha)}(\mathbb{C}^+) \quad \text{isometry}$$

whence $B_{m,\alpha} := W^{-\alpha}(z^{-\alpha}\mathcal{L}\ell_m)$ is OB, and

$$\begin{aligned} W^{-\alpha}(z^{-\alpha}\mathcal{L}\ell_m)(z) &= \frac{2}{\Gamma(\alpha)} \int_1^\infty \frac{(u-1)^{\alpha-1}}{u^\alpha} \frac{(2uz-1)^m}{(2uz+1)^{m+1}} du \\ &= \frac{(-1)^m}{2z}\mathcal{L}(\ell_{m,\alpha})(\frac{1}{4z}). \end{aligned}$$

Theorem

$$H_2^{(\alpha)}(\mathbb{C}^+) = H^2_{(\alpha)}(\mathbb{C}^+)$$

Proof. For all $z, w \in \mathbb{C}^+$,

$$\begin{aligned} K_{H^2_{(\alpha)}}(z,w) &= \sum_{j=0}^{\infty} B_{j,\alpha}(z)\overline{B_{j,\alpha}(w)} = \frac{1}{4z\overline{w}} \sum_{j=0}^{\infty} \mathcal{L}(\ell_{j,\alpha})(\frac{1}{4z})\overline{\mathcal{L}(\ell_{j,\alpha})(\frac{1}{4w})} \\ &= \frac{1}{4z\overline{w}} \int_0^1 \int_0^1 \frac{(1-x)^{\alpha-1}(1-y)^{\alpha-1}}{\Gamma(\alpha)^2} \frac{dx\, dy}{(x/4z) + (y/4\overline{w})} \\ &= \int_0^1 \int_0^1 \frac{(1-x)^{\alpha-1}(1-y)^{\alpha-1}}{\Gamma(\alpha)^2} \frac{dx\, dy}{x\overline{w} + yz} = K_{H_2^{(\alpha)}}(z,w) \end{aligned}$$

- The Laplace transform is an isometric isomorphism from $\mathcal{T}_2^{(\alpha)}(t^\alpha) \equiv \mathfrak{B}_\alpha$ onto $H_2^{(\alpha)}(\mathbb{C}^+) = H^2_{(\alpha)}(\mathbb{C}^+)$, where $F \in H_2^{(\alpha)}(\mathbb{C}^+)$ means: $F \in H_2(\mathbb{C}^+)$ and there exists $W^\alpha F$ holomorphic on $\mathbb{C}^+$ such that $z^\alpha W^\alpha F \in H_2(\mathbb{C}^+)$, $F = W_{rad}^{-\alpha}(W^\alpha F)$.

- Also,

$$H_2^{(\beta)}(\mathbb{C}^+) \overset{\beta > \alpha}{\hookrightarrow} H_2^{(\alpha)}(\mathbb{C}^+) \hookrightarrow H_2(\mathbb{C}^+).$$

ESTIMATING THE KERNEL

$$J_\theta := \int_0^1 \frac{1}{t^2+1+2t\cos 2\theta}\, dt = \frac{|\theta|}{|\sin 2\theta|}, \text{ if } \theta \neq 0; \quad J_0 = 1/2.$$

Then

$$K_\alpha(z,z) = \cdots = \frac{2\cos\theta}{|z|} \int_0^1 \int_0^1 \frac{(1-yt)^{\alpha-1}(1-y)^{\alpha-1}}{\Gamma(\alpha)^2} \frac{(1+t)\; dy\; dt}{t^2+1+2t\cos 2\theta}$$

so that

$$\frac{1}{(2\alpha-1)\Gamma(\alpha)^2}\,\frac{1}{|z|} \leq \|K_\alpha(\cdot,z)\|^2 \leq \frac{\pi}{(2\alpha-1)\Gamma(\alpha)^2}\,\frac{1}{|z|}, \quad \text{if } \alpha \geq 1,$$

and

$$\frac{1}{\Gamma(\alpha)^2}\,\frac{1}{|z|} \leq \|K_\alpha(\cdot,z)\|^2 \leq \frac{M_\alpha \pi}{(2\alpha-1)\Gamma(\alpha)^2}\,\frac{1}{|z|}, \quad \text{if } \frac{1}{2} < \alpha < 1,$$

Theorem

$$\|K_\alpha(\cdot,z)\|_{H_2^{(\alpha)}} \simeq \frac{1}{\sqrt{|z|}}, \quad \forall z \in \mathbb{C}^+.$$

GO FURTHER

Q.- *Composition operators.*

Take $\alpha = n \in \mathbb{N}$. Let $\varphi\colon \mathbb{C}^+ \to \mathbb{C}^+$ be holomorphic.

Characterize φ such that

$$C_\varphi\colon f \in H_2^{(n)}(\mathbb{C}^+) \mapsto f \circ \varphi \in H_2^{(n)}(\mathbb{C}^+)$$

is well defined.

(Work in progress, with M., S-L. and V. Matache)

Q.- Relation with self-similarity and de Branges spaces.

$$f \mapsto \sqrt{\lambda}\, f(\lambda\,\cdot), \quad \lambda > 0.$$

Q.- Role of Cesàro operators (boundedness, spectral properties, ...).

THANK YOU FOR YOUR ATTENTION

(see all tomorrow for dinner)