Multipliers between sub-Hardy Hilbert spaces

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Multipliers :

$$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) = \{\varphi \in \mathrm{Hol}(\mathbb{D}) : \varphi f \in \mathcal{H}_2, \text{ for all } f \in \mathcal{H}_1\}$$

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Classical examples :
Let H^∞ Hardy space of bounded analytic functions on $\mathbb{D}$, then

- $\mathcal{M}(H^2, H^2) = H^\infty$ (H^2 Hardy space)
- $\mathcal{M}(A^2, A^2) = H^\infty$ (A^2 Bergman space)
- $\mathcal{M}(\mathcal{D}, \mathcal{D}) \subsetneq H^\infty$ ($\mathcal{D}$ Dirichlet space) [1]

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Two situations :

- $\mathcal{H}_1, \mathcal{H}_2$ are model spaces.
- $\mathcal{H}_1, \mathcal{H}_2$ are the range of coanalytic Toeplitz operators equipped with the range norm.

Both situations can be viewed as examples of Hilbert spaces which are boundedly contained into H^2.

The multiplier's question in that context appear naturally e.g. in partial orders on partial isometries (see works of Garcia–Martin–Ross and Timotin).

Model spaces

u a non constant inner function : $u \in H^\infty$, $|u| = 1$ a.e. on $\mathbb{T}$.
Factorization : $u = BS_\mu$, with

- Blaschke product : $B = \prod_{\lambda \in \Lambda} b_\lambda$, $b_\lambda(z) = \dfrac{|\lambda|}{\lambda} \dfrac{\lambda - z}{1 - \overline{\lambda} z}$, $\sum_{\lambda \in \Lambda} (1 - |\lambda|^2) < +\infty$,
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Model space :

$$K_u = H^2 \ominus uH^2 = H^2 \cap u\overline{zH^2}$$

is the generic backward shift invariant : $S^*K_u \subset K_u$, $Sf(z) = zf(z)$

Examples 1

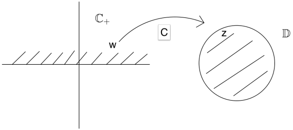
1. $K_{z^n} = \text{Pol}_{n-1}$
2. $K_B = \bigvee\{k_\lambda : \lambda \in \Lambda\}$, $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$, $\Lambda \subset \mathbb{D}$ Blaschke, no multiplicity.
3. K_{θ_a} is unitarily isomorphic to $PW_a = \mathcal{F}(L^2(-a,a))$, $\theta_a(z) = S_{2a\delta_{\{1\}}}(z) = e^{2a\frac{z+1}{z-1}}$, $a > 0$.

More precisely, let $C : \mathbb{C}_+ \longrightarrow \mathbb{D}$, $C(w) = z = \frac{w-i}{w+i}$.

Let H^2_+ be the Hardy space on $\mathbb{C}_+$. Then

$$\mathcal{U} : H^2 \longrightarrow H^2_+, \mathcal{U}f(w) = \frac{1}{\sqrt{\pi}(w+i)} f(C(w))$$

is unitary and $\mathcal{U}K_{\theta_a} = K_{\theta_a \circ C}$.
Note that $(\theta_a \circ C)(w) = e^{2iaw} = S^{2a}(w)$,
with $S(w) = e^{iw}$ and $K_{S^{2a}} = S^a PW_a$.



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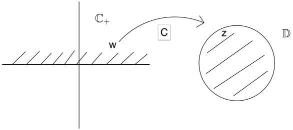

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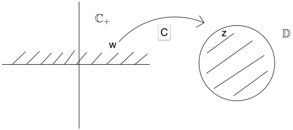

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Crofoot characterized onto multipliers in terms of arguments of u and v and a Carleson measure condition.

The Carleson measure condition is obviously necessary.

Indeed, let $\varphi \in \mathcal{M}(u,v)$. Then, $\varphi K_u \subset K_v \subset H^2$, that is

$$\int_{\mathbb{T}} |f(\zeta)|^2 |\varphi(\zeta)|^2 \, dm(\zeta) < \infty, \qquad (f \in K_u),$$

which means that $|\varphi|^2 dm$ has to be a Carleson measure for K_u.

Kernel of Toeplitz operators

Toeplitz operator : $T_\psi f = P_+(\psi f)$, where $\psi \in L^\infty(\mathbb{T})$,
$P_+(\sum_{n\in\mathbb{Z}} a_n e^{int}) = \sum_{n\geq 0} a_n e^{int}$ Riesz projection.

Observe that $K_v = \ker T_{\overline{v}}$, $S^*u \in K_u$ and $T_{1-u(0)\bar{u}}$ is invertible. Using these facts, along with the identity (on $\mathbb{T}$)

$$\bar{v}\varphi S^*u = \bar{v}\varphi\bar{z}(u-u(0)) = (1-u(0)\bar{u})\overline{zv}u\varphi,$$

it follows that $\varphi \in \mathcal{M}(u,v) \implies \varphi \in \ker T_{\overline{zv}u}$.

So to summary, if $\varphi \in \mathcal{M}(u,v)$, then

$$|\varphi|^2 dm \text{ has to be a Carleson measure for } K_u, \tag{1}$$

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Characterization of multipliers from one model space to another

Theorem 1 (F.-Hartmann-Ross, 2016)

For inner functions u and v and $\varphi \in H^2$, the following are equivalent :

1. $\varphi \in \mathcal{M}(u, v)$;
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Multiplier problem <——> two classical and difficult problems in function and operator theory :

1. Carleson measures for model spaces. Works by Cohn, Treil–Volberg, Baranov and recently by Lacey–Sawyer–Shen–Uriarte-Tuero–Wick....
2. Kernel of Toeplitz operators. Works by Sarason, Makarov–Poltoratski,....

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- Are there unbounded multipliers ?
- Crofoot's question : are there unbounded *onto* multipliers ?
- Are there more explicit characterizations of multipliers ?

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First evidence : if u and v are Blaschke products, then $\mathcal{M}(u,v) \neq \{0\}$ if and only if $d^o u \leq d^o v$ (degree=number of zeros). We have

Theorem 2 (F.-Hartmann-Ross 2015 ; Crofoot $n=m$ 1994)

u finite Blaschke product, zeros $\{a_1,\dots,a_m\}$; v finite Blaschke product, zeros $\{b_1,\dots,b_n\}$, $m \leqslant n$, then

$$\mathcal{M}(u,v) = \left\{ q(z) \frac{\prod_{i=1}^{m}(1-\overline{a_i}z)}{\prod_{j=1}^{n}(1-\overline{b_j}z)} : q \in \mathrm{Pol}_{n-m} \right\}.$$

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Corollary 3

If u is a FBP, $d^o u = n$, v is any inner function which is not a FBP. Then $\mathcal{M}(u,v) \cap H^\infty \neq \{0\}$.

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Pf. : Frostman shift : $\exists a \in \mathbb{D}$, $v_a = \frac{v-a}{1-\overline{a}v}$ infinite BP. Let B formed with n zeros from v_a. Then $\exists$ (onto) multipliers $\varphi : K_u \to K_B \subset K_{v_a}$. With Crofoot : $\varphi(1-\overline{a}v) : K_u \to K_v$. □

Bounded and unbounded multipliers

Let $\mathcal{H}$ be a RKHS on $\mathbb{D}$ with kernel function $k_\lambda^{\mathcal{H}}$ and $\varphi \in \mathcal{M}(\mathcal{H},\mathcal{H})$.

Using reproducing property, we easily see that

$$M_\varphi^* k_\lambda^{\mathcal{H}} = \overline{\varphi(\lambda)} k_\lambda^{\mathcal{H}},$$

and if $\forall \lambda \in \mathbb{D}$, $k_\lambda^{\mathcal{H}} \neq 0$, then $\sup_\lambda |\varphi(\lambda)| \lesssim \|M_\varphi^*\|$.
In particular, φ is bounded.

The situation changes when we consider $\mathcal{M}(u,v)$ for $u \neq v$!

Indeed, if k_λ^u, k_λ^v are the reproducing kernels for K_u, K_v respectively, then we have $M_\varphi^* k_\lambda^v = \overline{\varphi(\lambda)} k_\lambda^u$,

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For $u = BS_\mu$ inner, we define the boundary spectrum

$$\sigma(u) = \{\zeta \in \mathbb{T} : \liminf_{z \to \zeta} |u(z)| = 0\} = \operatorname{supp}(\mu) \cup (\mathbb{T} \cap B^{-1}(\{0\})).$$

Recall that every $f \in K_u$ can be extended analytically through $\mathbb{T} \setminus \sigma(u)$.

Theorem 4 (F-Hartmann-Ross, 2016)

Let u, I inner and suppose $\sigma(u) \cap \sigma(I) = \emptyset$. If $v = uI$, then $\mathcal{M}(u,v) = K_{zI}$. Furthermore, if I is not a finite Blaschke product, then $\mathcal{M}(u,v)$ contains unbounded functions.

Pf. : Observe that $\ker T_{\overline{zv}u} = \ker T_{\overline{zI}} = K_{zI}$. Hence, by Theorem 1, we just need to check that $|\varphi|^2\, dm$ is a Carleson measure for K_u for every $\varphi \in K_{zI}$.

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$$\sigma(u) = \{\zeta \in \mathbb{T} : \liminf_{z\to\zeta} |u(z)| = 0\} = \operatorname{supp}(\mu) \cup (\mathbb{T} \cap B^{-1}(\{0\})).$$

Recall that every $f \in K_u$ can be extended analytically through $\mathbb{T} \setminus \sigma(u)$.

Theorem 4 (F-Hartmann-Ross, 2016)

Let u, I inner and suppose $\sigma(u) \cap \sigma(I) = \emptyset$. If $v = uI$, then $\mathcal{M}(u, v) = K_{zI}$. Furthermore, if I is not a finite Blaschke product, then $\mathcal{M}(u, v)$ contains unbounded functions.

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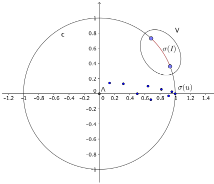
Let V be a neighborhood of $\sigma(I)$ far from $\sigma(u)$ and $\varphi \in K_{zI}$. Then φ extends analytically through $\mathbb{T}$ outside V, and can be assumed bounded there. Similarly every $f \in K_u$ extends analytically in V and can be assumed to be bounded there. Hence, for every $f \in K_u$, we have



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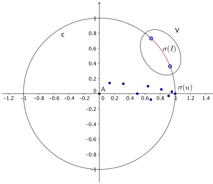

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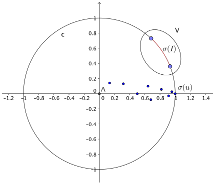

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A situation where all multipliers are bounded

Theorem 5 (F-Hartmann-Ross, 2016)

Let u, v inner. If, for some $\varepsilon_1, \varepsilon_2 \in (0,1)$, $\{|v| < \varepsilon_2\} \subset \{|u| < \varepsilon_1\}$, then $\mathcal{M}(u,v) = \ker T_{\overline{zv}u} \cap H^\infty$.

The proof is based on the following result of Cohn :
Let θ be an inner function and $f \in K_\theta$. If f is bounded on $\{|\theta| < \varepsilon\}$ for some $\varepsilon \in (0,1)$, then $f \in H^\infty$.

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A situation when the Carleson condition becomes more tractable : Let U be an inner in $\mathbb{C}_+$ such that $|U'(x)| \asymp 1$, $x \in \mathbb{R}$. Let μ be a Borel measure on $\mathbb{R}$. Then, Baranov proved that μ is a Carleson measure for K_U if and only if $\sup_{x\in\mathbb{R}} \mu([x, x+1]) < \infty$.

Theorem 6 (F-Hartmann-Ross, 2016)

Let U, V inner in $\mathbb{C}_+$ with $|U'(x)| \asymp 1$, $x \in \mathbb{R}$. Then

$$\mathcal{M}(U,V) = \left\{ \Phi \in (z+i) \ker T_{\overline{b_i^+ V} U} : \sup_{x\in\mathbb{R}} \int_x^{x+1} |\Phi(t)|^2 \, dt < \infty \right\}.$$

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An answer to Crofoot's question

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Non triviality of $\ker T_{\overline{V}U}$.

Assume that U, V are inner functions in $\mathbb{C}_+$ and $|U'(x)| \asymp 1$, $x \in \mathbb{R}$. Let $\Phi \in \ker T_{\overline{V}U}$. Then, there is a function $g \in H^2_+$ such that $\Phi(x)U(x)\overline{V(x)} = \overline{g(x)}$, $x \in \mathbb{R}$.

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which gives $\Phi \in (z+i)\ker T_{\overline{Vb_i^+}U}$.

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We denote by $L^1_\Pi = L^1(\mathbb{R}, \Pi)$, where Π is the Poisson measure on $\mathbb{R}$, that is $d\Pi(t) = \frac{dt}{1+t^2}$. Recall also that the Hilbert transform of a function $h \in L^1_\Pi$ is defined as the singular integral

$$\tilde{h}(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) h(t)\, dt.$$

Theorem 7 (F-Rupam, 2018)

Let U, V be MIF with $|U'(x)| \asymp 1$, $x \in \mathbb{R}$, and let $m := \arg(U) - \arg(V b_i^+)$ on $\mathbb{R}$. Assume that either $m \notin \widetilde{L^1_\Pi}$ or if $m = \tilde{h}$ for some $h \in L^1_\Pi$, then $e^{-h} \notin L^1$. TFAE :

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Example 1

Let $U = B_\Lambda S^a$ and $V = S^b$, where $a, b \geq 0$ and B_Λ is an infinite BP. Assume that $|U'(x)| \asymp 1$, $x \in \mathbb{R}$ and $b - a \neq 2\pi D$. Then

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The range spaces of co analytic Toeplitz operators

Let $a \in H^\infty$ outer and denote by $\mathfrak{M}(\bar{a}) = T_{\bar{a}} H^2$ equipped with the range norm

$$\|T_{\bar{a}} f\|_{\bar{a}} = \|f\|_2, \qquad f \in H^2.$$

Note that $\mathfrak{M}(\bar{a})$ are RKHS which are boundedly contained in H^2. They are closely connected to de Branges–Rovnyak spaces.

Given $a_1, a_2 \in H^\infty$ outer, study

$$\mathcal{M}(a_1, a_2) := \{\varphi \in \mathcal{O}(\mathbb{D}) : \forall f \in \mathfrak{M}(\overline{a_1}), \varphi f \in \mathfrak{M}(\overline{a_2})\}.$$

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Note that $1 \in \mathfrak{M}(\overline{a})$. Thus, if $\varphi \in \mathcal{M}(a)$, we get that $\varphi \in \mathfrak{M}(\overline{a})$. Using an argument with reproducing kernel, we also see that $\varphi \in H^\infty$. Thus

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The class $\mathcal{A}$

We define $\mathcal{A}$ as the class of polynomials whose all the zeros lies on $\mathbb{T}$. We proved that if $a \in \mathcal{A}$, then

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Questions :

1. Characterize a such that $\mathfrak{M}(\overline{a}) \cap H^\infty$ is an algebra.
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2. When $\mathfrak{M}(\overline{a}) \cap H^\infty$ is an algebra, does it imply that necessarily $\mathcal{M}(a) = \mathfrak{M}(\overline{a}) \cap H^\infty$?

The class $\mathcal{A}$

We define $\mathcal{A}$ as the class of polynomials whose all the zeros lies on $\mathbb{T}$. We proved that if $a \in \mathcal{A}$, then

$$\mathfrak{M}(\overline{a}) = aH^2 \oplus \mathrm{Pol}_{N-1},$$

where N is the degree of a.

Theorem 9 (F.-Hartmann–Ross, 2018)

Let $a \in \mathcal{A}$. Then

$$\mathcal{M}(a) = \mathfrak{M}(\overline{a}) \cap H^\infty.$$

Cons. : if $a \in \mathcal{A}$, then $\mathfrak{M}(\overline{a}) \cap H^\infty$ is an algebra ! This is not true in general (Sarason and Lotto constructed an example).

Questions :

1. Characterize a such that $\mathfrak{M}(\overline{a}) \cap H^\infty$ is an algebra.
2. When $\mathfrak{M}(\overline{a}) \cap H^\infty$ is an algebra, does it imply that necessarily $\mathcal{M}(a) = \mathfrak{M}(\overline{a}) \cap H^\infty$?

Theorem 10 (F.-Hartmann–Ross, 2018)

Let $a_1, a_2 \in \mathcal{A}$.

1. *If* $h = a_1/a_2 \in H^\infty$, *then*
$$\mathcal{M}(a_1, a_2) = \{\varphi \in \mathfrak{M}(\overline{a_2}) : h\varphi \in H^\infty\}.$$
2. *If* $k = a_2/a_1 \in H^\infty$, *then*
$$\mathcal{M}(a_1, a_2) = k(\mathfrak{M}(\overline{a_1}) \cap H^\infty).$$

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Merci pour votre attention !

Thank you for your attention !