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"On Sack's Lemma"

Joint work with Oli Roth
(Wuerzburg)

Lemma of Jack (SMOOTH)

$F \in H(\bar{\mathbb{D}})$, $\quad |F(\xi)| = \max_{|z| \le 1} |F(z)| = \max_{|z| = |\xi| = 1} |F(z)| > 0$

Then

$$\frac{\xi F'(\xi)}{F(\xi)} \geq 0 \quad \text{with equality only if } F \equiv K$$

Clunie-Jack 1971

✓ Loewner (1930), ✓ Pólya-Szegö (1926)

✓ Hayman (1950)

✓ Erdos (Bak-Newman 1990)

✓ Miller-Mocanu (1980) Ruscheweyh (1980)

✓ Osserman (2000)

✓ Dubinin-Kim (2002)

✓ H. P. Boas (2010)

✓ Krantz (2010) ...

✓ Fournier (2001)

Lemma of Jack (LESS SMOOTH)

$F \in H(\mathbb{D})$, $F(\mathbb{D}) \subseteq \mathbb{D}$, $\zeta \in \partial\mathbb{D}$. The following are equivalent:

✓ ① $\liminf_{z \to \zeta} \dfrac{1 - |F(z)|}{1 - |z|} < \infty$

✓ ② $\lim_{\substack{r \to 1 \\ r<1}} F(r\zeta) := F(\zeta)$ and $\lim_{\substack{r \to 1 \\ r<1}} F'(r\zeta) := F'(\zeta)$ exist (finite)

$|F(\zeta)| = 1$

✓ ③ $\lim_{\substack{r \to 1 \\ r<1}} \dfrac{F(\zeta) - F(r\zeta)}{\zeta - r\zeta} = \lim_{\substack{r \to 1 \\ r<1}} F'(r\zeta) := F'(\zeta)$ exist (finite)

$|F(\zeta)| = 1$

Moreover ④ $0 < \zeta \dfrac{F'(\zeta)}{F(\zeta)}$

∴ one proof is based on the Julia-Carathéodory Theorem which also yields

$$\lim_{\substack{r \to 1 \\ r<1}} \frac{1 - |F(r\zeta)|}{1 - r} = \lim_{\substack{r \to 1 \\ r<1}} \frac{1 - F(r\zeta)/F(\zeta)}{1 - r} = \zeta \frac{F'(\zeta)}{F(\zeta)}$$

∴ the strict positivity in ④ also follows from the lemma of Hopf (as explained by T. Ransford in his book)

The following result can be obtained without using the Julia-Carathéodory Theorem or even the lemma of Julia

Jack's lemma:

$F \in H(\mathbb{D})$, $F(\mathbb{D}) \subseteq \mathbb{D}$, $\xi \in \partial\mathbb{D}$ with ✓ $\lim_{\substack{0<r<1 \\ r\to 1}} F(r\xi) := F(\xi)$, $|F(\xi)|=1$

✓ $\lim_{\substack{0<r<1 \\ r\to 1}} F'(r\xi) := F'(\xi)$ (finite)

Then $\boxed{\xi \dfrac{F'(\xi)}{F(\xi)} > 0}$

Hint: use $\dfrac{1+F(z)}{1-F(z)} = \displaystyle\int_0^{2\pi} \dfrac{1+e^{-i\theta}z}{1-e^{-i\theta}z}\, d\mu(\theta)$

Julia's lemma: $F \in H(\mathbb{D})$, $F(\mathbb{D}) \subseteq \mathbb{D}$, $\xi \in \partial\mathbb{D}$, $\liminf_{z\to\xi} \dfrac{1-|F(z)|}{1-|z|} < \infty$

then $\dfrac{|1-F(z)\overline{F(\xi)}|^2}{1-|F(z)|^2} \dfrac{1-|z|^2}{|1-z\bar{\xi}|^2} \leq \xi \dfrac{F'(\xi)}{F(\xi)}$, $z \in \mathbb{D}$

with equality for automorphisms of the disc.

In what follows, I shall prove

Jack's lemma $\Rightarrow$ Julia's lemma

(and more!)

Let $\{z_k\} \subset \mathbb{D}$, $f : \mathbb{D} \to \mathbb{D}$, $\xi \in \partial \mathbb{D}$ with

$$(*) \qquad \liminf_{z \to \xi} \frac{1-|f(z)|}{1-|z|} < \infty .$$

Set $$f_{k+1}(z) = \frac{1-\bar{z}_k z}{z-z_k} \, \frac{f_k(z)-f_k(z_k)}{1-\overline{f_k(z_k)}\, f_k(z)} , \quad k \geq 0$$

$$f_0 \equiv f$$

Each function f_k satisfies $(*)$ with

$$\frac{\xi f_{k+1}'(\xi)}{f_{k+1}(\xi)} = \frac{\xi f_k'(\xi)}{f_k(\xi)} \, \frac{1-|f_k(z_k)|^2}{|1-\overline{f_k(z_k)}\, f_k(\xi)|^2} - \frac{1-|z_k|^2}{|1-\bar{z}_k \xi|^2}$$

and by <u>Jack</u>

$$0 \leq \frac{\xi f_1'(\xi)}{f_1(\xi)} = \frac{\xi f'(\xi)}{f(\xi)} \, \frac{1-|f(z_0)|^2}{|1-\overline{f(z_0)}\, f(\xi)|^2} - \frac{1-|z_0|^2}{|1-\bar{z}_0 \xi|^2}$$

i.e. $$\frac{\xi f'(\xi)}{f(\xi)} \geq \frac{|1-\overline{f(z_0)}\, f(\xi)|^2}{1-|f(z_0)|^2} \, \frac{1-|z_0|^2}{|1-\bar{z}_0 \xi|^2}$$

(<u>Julia</u>!)

Iteration of this procedure shall yield ($n \geq 0$)

$$\boxed{\frac{\xi f'(\xi)}{f(\xi)} \geq \sum_{j=0}^{n} \left(\prod_{k=0}^{j} \frac{|1 - \overline{f_k(z_k)}\, f(\xi)|^2}{1 - |f_k(z_k)|^2} \right) \frac{1 - |z_j|^2}{|1 - \overline{z_j}\, \xi|^2}}$$

with equality if and only if f is a Blaschke product of order $n+1$

(some sort of "multi point" lemma of Julia!)

In particular

$$\sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \frac{|1 - \overline{f_k(z_k)}\, f(\xi)|^2}{1 - |f_k(z_k)|^2} \right) \frac{1 - |z_j|^2}{|1 - \overline{z_j}\, \xi|^2}$$

is convergent ! ?

Two special cases

i) take $z_k = 0$, $k = 0, 1, 2, \ldots$

$$\frac{\xi f'(\xi)}{f(\xi)} \geq \sum_{j=0}^{n} \left(\prod_{k=0}^{j} \frac{|1 - \overline{f_k(0)}\, f(\xi)|^2}{1 - |f_k(0)|^2} \right)$$

$$\geq \sum_{j=0}^{n} \prod_{k=0}^{j} \frac{1 - |f_k(0)|}{1 + |f_k(0)|}$$

$n = 0$ Osserman (2000)

$n = 1$ Lecko and Uzar (2013)

a lower bound depending on $|a_0|, |a_1|, \ldots |a_n|$ where

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad z \in \mathbb{D}$$

in particular $\sum_{j=0}^{\infty} \prod_{k=0}^{j} \frac{1 - |f_k(0)|}{1 + |f_k(0)|}$ is convergent and

$$\lim_{j \to \infty} j \prod_{k=0}^{j} \frac{1 - |f_k(0)|}{1 + |f_k(0)|} = 0 \quad ?$$

ii) Let $F: \mathbb{D} \to \mathbb{D}$ and $\{z_j\}$ the zeros of F in $\mathbb{D}$ and also assume that $\xi \in \partial\mathbb{D}$ and

$\liminf\limits_{z \to \xi} \dfrac{1 - |F(z)|}{1 - |z|} < \infty$. Define

$$F_{j+1}(z) = F(z) \prod_{k=0}^{j} \frac{1 - \bar{z}_k z}{z - z_k}$$

Then

$$\frac{\xi F_{j+1}'(\xi)}{F_{j+1}(\xi)} = \frac{\xi F'(\xi)}{F(\xi)} - \sum_{k=0}^{j} \frac{1 - |z_k|^2}{|1 - \bar{z}_k \xi|^2} \geq 0$$

i.e. under the hypothesis, $\sum\limits_{k=0}^{\infty} \dfrac{1 - |z_k|^2}{|1 - \bar{z}_k \xi|^2}$ converges $\leq \dfrac{\xi F'(\xi)}{F(\xi)}$

note: in the case where F is a Blaschke product, the convergence of $\sum\limits_{k=0}^{\infty} \dfrac{1 - |z_k|^2}{|1 - \bar{z}_k \xi|^2}$ is necessary and sufficient for the existence of $F(\xi)$ and $F'(\xi)$. Further $\dfrac{\xi F'(\xi)}{F(\xi)} = \sum\limits_{k=0}^{\infty} \dfrac{1 - |z_k|^2}{|1 - \bar{z}_k \xi|^2}$. This is a result of Frostman (1945)

Known result:

$F \in H(\mathbb{D} \cup \{\xi\})$, $|\xi| = 1$, $|F(\xi)| = \max\limits_{|z| \le 1} |F(z)| > 0$.

Then $\mathrm{Re}\left(1 + \xi \dfrac{F''(\xi)}{F'(\xi)}\right) \geq \xi \dfrac{F'(\xi)}{F(\xi)} \geq 0$

I have a proof that equality holds here if and only if $|F(z)|$ is constant for all $z \in \partial\mathbb{D}$ close enough to ξ.

What about the non-smooth case?
Which extra hypotheses are needed?