Exotic Ideals in the Fourier-Stieltjes Algebra of a Locally Compact group.

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The Players: L_p Fourier and Fourier-Stieltjes Algebras

Set up:

- G-locally compact group (LCG) with fixed left Haar measure μ.

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- Σ_G = all equivalence classes of continuous unitary reps.

Definition (Group C^*-algebra)

For $f \in L^1(G)$ define

$$\|f\|_u = \sup\{\|\pi(f)\|_{B(\mathcal{H}_\pi)} | \pi \in \Sigma_G\}$$

and

$$C^*(G) = \text{ completion of } L^1(G) \text{ wrt } \|\cdot\|_u$$

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Definition ($C^*_\pi(G)$-algebra)

Let $\pi \in \Sigma_G$. Let $\mathcal{N}_\pi$ be the kernel of $\pi : C^*(G) \to B(\mathcal{H}_\pi)$. Then

$$C^*_\pi(G) = C^*(G)/\mathcal{N}_\pi$$

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Example

The left-regular representation $\lambda : G \to \mathcal{U}(L^2(G))$ is defined as

$$(\lambda(x)f)(y) = f(x^{-1}y).$$

$C^*_\lambda(G)$ is called the reduced C^*-algebra of G.

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Theorem (Hulanicki, 1964)

$C^*(G) = C^*_\lambda(G)$ if and only if G is amenable.

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Question: If G is non-amenable do there exist intermediate C^*-algebras between $C^*(G)$ and $C^*_\lambda(G)$?

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Definition (L_p-representations)

π is an L_p representation for $1 \leq p < \infty$ if

$$\pi_{\xi,\xi}(x) := \langle \pi(x)\xi, \xi \rangle \in L_p(G)$$

for a dense set of vectors $\xi \in \mathcal{H}_\pi$.

$$\pi_p = \{\oplus\pi |\ \pi \text{ is an } L_P \text{ representation}\}$$

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Theorem (Brown and Guentner: Okayasu)

If $G = \mathbb{F}_2$ and $2 < p < q < \infty$, then $C^*_\lambda(G)$, $C^*_{\pi_p}(G)$, $C^*_{\pi_q}(G)$ and $C^*(G)$ are all distinct.

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Definition (Eymard: Fourier-Stieltjes Algebra of G)

$$B(G) = \{\pi_{\xi,\eta} | \pi \in \Sigma_G\}$$

with

$$\|f\|_{B(G)} = min\{\|\xi\|\|\eta\| \, | u = \pi_{\xi,\eta}\}.$$

is a commutative Banach algebra called the Fourier-Stieltjes algebra of G.

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Key Facts:

1) $B(G) = (C^*(G))^*$

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Key Facts:

1) $B(G) = (C^*(G))^*$
2) $B(G)$ completely determines G.

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Definition (Arsac: $A_\pi(G)$ and $B_\pi(G)$)

Given $\pi \in \Sigma_G$, let

$$A_\pi(G) = span\{\pi_{\xi,\eta} | \xi, \eta \in \mathcal{H}_\pi\}^{-\|\cdot\|_{\mathcal{B}(\mathcal{G})}}$$

and

$$B_\pi(G) = A_\pi(G)^{-w^*} = (C^*_\pi(G))^*$$

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Example

For $1 \leq p \leq 2$

$$A_{\pi_p}(G) = A_\lambda(G) = A(G)$$

is called the Fourier Algebra of G.

$A(G)$ is a closed ideal of $B(G)$ with $\Delta(A(G)) = G$.

The Players: L_p Fourier and Fourier-Stieltjes Algebras

Note: For $G = \mathbb{F}_2$ and $1 < p < \infty$, the $B_{\pi_p}(G)$'s are distinct closed ideals in $B(G)$ containing $A(G)$, but for G amenable

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Question: What can we say about $A_{\pi_p}(G)$ for $2 < p < \infty$?

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Question: What can we say about $A_{\pi_p}(G)$ for $2 < p < \infty$?

Theorem (Wiersma)

For $2 \leq p \leq \infty$, $A_{\pi_p}(G)$ is a closed ideal of $B(G)$ with $\Delta(A_\pi(G)) = G$ containing $A(G)$ which completely determines G .

$A_{\pi_p}(G))$ for $2 \leq p \leq \infty$

Note:

1) For $G = \mathbb{F}_2$, we have that for $2 < p < q < \infty$ that

$$A(G) \subsetneq A_{\pi_p}(G) \subsetneq A_{\pi_q}(G) \subsetneq B_0(G)$$

where $B_0(G) = B(G) \cap C_0(G)$ is the Rajchman algebra of G.

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2) If G is compact then

$$A(G) = A_{\pi_p}(G) = B(G)$$

for all $1 \leq p < \infty$.

Question: For which locally compact groups are the $A_{\pi_p}(G)$ ideals distinct for $2 < p < \infty$?

$A_{\pi_p}(G)$ for $2 \leq p \leq \infty$

Theorem (Wiersma)

If G is non-compact and abelian, then for $2 < p < q < \infty$ *we have that*

$$A(G) \subsetneq A_{\pi_p}(G) \subsetneq A_{\pi_q}(G) \subsetneq B_0(G)$$

$A_{\pi_p}(G)$ for $2 \leq p < \infty$

Theorem (F, Tanko, Wiersma)

If G is an infinite discrete group which satisfies one of the following conditions:

1) *G is locally finite,*
2) *G is elementary amenable,*
3) *G is a linear group,*
4) *G has polynomial growth,*

then for $2 < p < q < \infty$ we have

$$A(G) \subsetneq A_{\pi_p}(G) \subsetneq A_{\pi_q}(G) \subsetneq B_0(G)$$

and $A_{\pi_p}(G)$ is non-separable.

$A_{\pi_p}(G)$ for $2 \leq p < \infty$

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and $A_{\pi_p}(G)$ is non-separable.

Question: Are the $A_{\pi_p}(G)$ ideals distinct for $2 < p < q < \infty$ for every infinite discrete group?

$A_{\pi_p}(G)$ for $2 \leq p < \infty$

Definition

1) G is [IN] is G has a neighborhood of the identity that is invariant under inner automorphisms. ,
2) G is [MAP] if the finite dimensional reps. separate points.
3) G almost connected if G/G_0 is compact.

$A_{\pi_p}(G)$ for $2 \le p < \infty$

Definition

1) G is [IN] is G has a neighborhood of the identity that is invariant under inner automorphisms. ,
2) G is [MAP] if the finite dimensional reps. separate points.
3) G almost connected if G/G_0 is compact.

Theorem (F, Tanko, Wiersma)

If G is a non-compact almost connected [IN]-group, then then for $2 < p < q < \infty$ we have

$$A(G) \subsetneq A_{\pi_p}(G) \subsetneq A_{\pi_q}(G) \subsetneq B_0(G)$$

and $A_{\pi_p}(G)$ is non-separable.

$A_{\pi_p}(G)$ for $2 \leq p < \infty$

Theorem (F, Tanko, Wiersma)

If G is either an [IN]-group or a [MAP]-group and if either

1) $A_{\pi_p}(G) = A_{\pi_q}(G)$ *for some* $2 < p < q$ *or*
2) $A_{\pi_p}(G)$ *is separable,*

then G has a compact, open subgroup.

$A_{\pi_p}(G)$ for $2 \leq p < \infty$

Question: If G is non-compact and $2 < p < q < \infty$ must

$$A(G) \subsetneq A_{\pi_p}(G) \subsetneq A_{\pi_q}(G) \subsetneq B_0(G)?$$

$A_{\pi_p}(G)$ for $2 \leq p < \infty$

Question: If G is non-compact and $2 < p < q < \infty$ must

$$A(G) \subsetneq A_{\pi_p}(G) \subsetneq A_{\pi_q}(G) \subsetneq B_0(G)?$$

Example

If

$$G = \{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a \neq 0, b \in \mathbb{R} \}$$

is the $ax + b$ group then for any $1 \leq p < \infty$, we have

$$A(G) = A_{\pi_p}(G) = B_0(G)$$

$A_{\pi_p}(G)$ for $2 \leq p < \infty$

For Garth!!!

Theorem (F, Tanko, Wiersma)

1) *$A_{\pi_p}(\mathbb{F}_2)$ is not BSE for any $1 \leq p < \infty$*

$A_{\pi_p}(G)$ for $2 \leq p < \infty$

For Garth!!!

Theorem (F, Tanko, Wiersma)

1) *$A_{\pi_p}(\mathbb{F}_2)$ is not BSE for any* $1 \leq p < \infty$
2) $A_*(\mathbb{F}_2) = (\bigcup_{2<p<\infty} A\pi_p(\mathbb{F}_2))^{-\|\cdot\|_{B(\mathbb{F}_2)}}$ *is BSE.*

THANK YOU