Factorization in commutative Banach algebras

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Commutative Banach algebras

Throughout, we shall be concerned with **commutative** Banach algebras = CBAs (always associative and over $\mathbb{C}$).

In particular, we shall think about:

• semi-simple CBAs, equivalently (natural) Banach function algebras (= BFAs) on a locally compact space;

• maximal ideals in (unital) uniform algebras - they are closed, unital subalgebras of $(C(X), |\cdot|_X)$, X compact, that separate the points of X;

• commutative, radical Banach algebras = CRBAs.

We are interested to know if the results are different if we restrict to separable examples.

We shall list a number of properties related to factorization; each implies the next; we are trying to give counter-examples to all reverse implications (in various classes of CBAs).

Some notation

Let A be a BFA on a locally compact space X, and take $x \in X$. Then ε_x is the evaluation functional $\varepsilon_x : f \mapsto f(x)$ and M_x is the corresponding maximal modular ideal; A is **natural** if all characters are evaluation functionals.

Let A be a BFA on X. Then $x \in X$ is a **strong boundary point** = SBP if, for each open neighbourhood U of x, there exists $f \in A$ with $f(x) = |f|_X = 1$ and $|f|_{X \setminus U} < 1$ (includes **peak points**).

For a compact plane set X, take $R(X)$ to be the uniform closure of the algebra of rational functions restricted to X; it is a natural uniform algebra on X.

Approximate identities

Let A be a CBA. Then an **approximate identity** (AI) for A is a net (e_ν) in A such that $\lim_\nu e_v a = a$ for each $a \in A$; the AI (e_ν) is a **bounded approximate identity** (BAI) if $\sup_\nu \|e_v\| < \infty$, and a **contractive approximate identity** (CAI) if $\|e_v\| \leq 1$ for each ν.

(I) Let A be a CBA. Then A has property (I) if A has a BAI.

Approximate identities for uniform algebras

We can characterize when some maximal ideals have a BAI.

Proposition Let A be a natural uniform algebra on a compact space X, and take $x \in X$. Then the following conditions on x are equivalent:

(a) x is a strong boundary point;

(b) M_x has a BAI;

(c) M_x has a CAI. $\square$

A natural uniform algebra on compact X is a **Cole algebra** if all points of X are SBPs. Such algebras not equal to $C(X)$ exist.

Null sequences

Let E be a Banach space. Then a **null sequence** in E is a sequence (x_n) in E such that $\lim_{n\to\infty} \|x_n\| = 0$; the space of null sequences in E is $c_0(E)$, and $c_0(E)$ is itself a Banach space for the norm defined by

$$\|(x_n)\| = \sup\{\|x_n\| : (x_n) \in c_0(E)\}\,.$$

Let A be a CBA. Then **null sequences factor** in A if, for each null sequence (a_n) in A, there exist $a \in A$ and a null sequence (b_n) in A such that $a_n = ab_n \ (n \in \mathbb{N})$.

[Important in automatic continuity theory - see my book.]

(II) Let A be a CBA. Then A has property (II) if all null sequences in A factor.

Cohen's factorization theorem

The following result is one form of the famous **Cohen's factorization theorem**; (much) more general forms are given in my book.

Theorem Let A be a CBA with a bounded approximate identity. Then null sequences factor, and so (I) $\Rightarrow$ (II) for A. $\square$

Reverse implication? **George Willis** (PLMS 1992) gave a separable BFA satisfying (II), but not (I), and this example can be modified to also give a separable CRBA with the same property.

New Theorem There is a maximal ideal in a uniform algebra satisfying (II), but not (I).

But our example is not separable - an open point. Proof later.

Factorization of pairs

Let A be a commutative algebra. Then **pairs factor** in A if, for each $a_1, a_2 \in A$, there exist $a, b_1, b_2 \in A$ such that $a_1 = ab_1$ and $a_2 = ab_2$.

(III) Let A be a CBA. Then A has property (III) if all pairs in A factor.

Trivially (II) $\Rightarrow$ (III).

But we cannot yet find any CBA such that pairs factor, but null sequences do not. Ugh.

Factorization

Let A be an algebra. Then:

$$A^{[2]} = \{ab : a, b \in A\}, \quad A^2 = \text{lin}\, A^{[2]}.$$

Definition The algebra A **factors** if $A = A^{[2]}$ and A **factors weakly** if $A = A^2$.

(IV) Let A be a CBA. Then A has property (IV) if A factors.

Trivially (III) $\Rightarrow$ (IV). Reverse implication?

Look at $H^\infty(\mathbb{D})$. It was shown by Ouzomgi (another nephew of Tom Ransford) that there is a maximal ideal M_x in $H^\infty(\mathbb{D})$ that factors, but such that pairs do not factor.

We would like a separable BFA and/or a CRBA that factors, but such that null sequences do not. Not known to us yet.

Weak factorization

(V) Let A be a CBA. Then A has property (V) if A factors weakly.

Trivially (IV) $\Rightarrow$ (V).

For a natural BFA on a compact X, this means that there are no non-zero point derivations at any point of X.

George Willis (PLMS 1992) gave a separable BFA such that every element in A is the sum of two products, and so A factors weakly, but such that A does not factor.

Theorem (**Rick Loy**) Let A be a **separable** Banach algebra that factors weakly. Then there exist $m \in \mathbb{N}$ and $M > 0$ such that each $a \in A$, has the form $a = \sum_{i=1}^{m} b_i c_i$, where $\sum_{i=1}^{m} \|b_i\| \|c_i\| \leq M \|a\|$. $\square$

In all known examples, we can take $m = 1$ or $m = 2$.

Weak factorization for uniform algebras

Embarrassing open point: we have no maximal ideal M in a uniform algebra that factors weakly, but does not factor. Any ideas?

[It is easy to get $M^{[2]} \neq M^2$, but we also want $M^2 = M$.]

Proposition Let X be a compact plane set. Then the following conditions on $x \in X$ with respect to the uniform algebra $R(X)$ are equivalent:

(a) x is a peak point;

(b) M_x has a BAI or CAI;

(c) M_x factors;

(d) M_x factors weakly. $\square$

So no counter-examples for $R(X)$.

Projective factorization

Let A be a BA, with projective tensor product $A \widehat{\otimes} A$. There is a unique bounded linear operator $\pi_A : A \widehat{\otimes} A \to A$ with $\pi_A(a \otimes b) = ab$ for $a, b \in A$, and then $\pi_A(A \widehat{\otimes} A)$ is a subalgebra of A and a Banach algebra with respect to the quotient norm from $(A \widehat{\otimes} A, \|\cdot\|_\pi)$.

Let A be a BA. Then A **factors projectively** if the map $\pi_A : A \widehat{\otimes} A \to A$ is a surjection, so that each $a \in A$ has the form $\sum_{i=1}^{\infty} b_i c_i$, where $b_i, c_i \in A$ and $\sum_{i=1}^{\infty} \|b_i\| \, \|c_i\| < \infty$.

(VI) Let A be a CBA. Then A has property (VI) if A factors projectively.

Trivially (V) $\Rightarrow$ (VI).

Easy counter-example to the reverse implication: $A = \ell^1$, with pointwise product.

What about a maximal ideal in a uniform algebra? What about a CRBA? Not known to us yet.

Examples of projective factorization

Here $\mathbb{Q}^{+\bullet} = \{r \in \mathbb{Q} : r > 0\}$, ω is a weight on $\mathbb{R}^+$, and so $\ell^1(\mathbb{Q}^{+\bullet}, \omega)$ is a CBA with respect to convolution multiplication.

At least we have the following:

Example Take $A = \ell^1(\mathbb{Q}^{+\bullet}, \omega)$ for a continuous weight ω on $\mathbb{R}^+$ which may be radical. Then A factors projectively, but pairs do not factor. Surely A does not factor? $\square$

Projective factorization in uniform algebras

Let A be a natural uniform algebra on compact X, and take $x, y \in X$. Then $x \sim y$ if $\|\varepsilon_x - \varepsilon_y\| < 2$. This is an equivalence relation on X, and the equivalence classess are the **Gleason parts** (wrt A).

These parts form a partition of X, and each part is a completely regular and σ-compact topological space with respect to the Gel'fand topology; by a theorem of **Garnett**, these are the only topological restrictions on Gleason parts.

Let A be a natural BFA on K. A net (e_α) in A is a **pointwise approximate identity** (PAI) if

$$\lim_\alpha e_\alpha(x) = 1 \quad (x \in K)\,;$$

the PAI (e_α) is **contractive** if $\sup_\alpha \|e_\alpha\| \leq 1$; we obtain a CPAI.

Proposition (**D-Ulger**) A maximal ideal M_x in a uniform algebra has a CPAI iff $\{x\}$ is a one-point part. $\square$

Projective factorization for $R(X)$

Proposition Let X be a compact plane set. Then the following conditions on $x \in X$ with respect to the uniform algebra $R(X)$ are equivalent:

(a) x is a peak point;

(b) $\{x\}$ is a one-point Gleason part;

(c) M_x has a CPAI;

(d) x is an isolated point with respect to the Gleason metric;

(e) M_x factors projectively. $\square$

So no counter-examples for $R(X)$.

The big disc algebra - 1

Take an irrational number α with $0 < \alpha < 1$, and consider the 'open half-plane' H_α consisting of points $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ with $m + n\alpha > 0$.

Then consider monomials on $\mathbb{C}^2$ of the form $Z^m W^n$, where $(m, n) \in H_\alpha$; here Z and W are the coordinate functionals on $\mathbb{C}^2$. We take $\mathfrak{A}_{0,\alpha}$ to be the linear span of these monomials and the constant function 1, and $\mathfrak{A}_\alpha$ to be the uniform closure of this algebra, regarded as a subalgebra of $C(\mathbb{T}^2)$. Then $\mathfrak{A}_\alpha$ is a separable uniform algebra on its character space Φ_α that can be identified with the space $\mathbb{T}^2 \times [0, 1]$, with the subset $\mathbb{T}^2 \times \{0\}$ identified to a point, called x_0; the corresponding maximal ideal in $\mathfrak{A}_\alpha$ at x_0 is denoted by $\mathfrak{M}_\alpha$. The set $\{x_0\}$ is a one-point part, but x_0 is not a peak point. (See the book of **Lee Stout** for all this.)

The big disc algebra - 2

Proposition The maximal ideal $\mathfrak{M}_\alpha$ of the big disc algebra factors projectively.

Proof For this, we use the fact that it follows from Dirichlet's theorem on Diophantine approximation that, for each $\varepsilon > 0$, there exist $p, q \in \mathbb{N}$ with

$$\alpha - \frac{\varepsilon}{q} < \frac{p}{q} < \alpha \,. \qquad \square$$

Hence (VI) $\not\Rightarrow$ (I) in the class of separable maximal ideals in uniform algebras.

We believe that there are null sequences in $\mathfrak{M}_\alpha$ that do not factor, but so far have not proved this.

Dense factorization

Let A be a CBA. Then A **factors densely** if A^2 is dense in A.

For a natural BFA on a compact X, this means that there are no non-zero, continuous point derivations at any point of X.

(VII) Let A be a CBA. Then A has property (VII) if A factors densely.

Trivially (VI) $\Rightarrow$ (VII).

Example Consider $R = C_{*,0}(\mathbb{I})$, the algebra of all continuous functions on $\mathbb{I}$ that vanish at 0, taken with the convolution product. Then R is a CRBA. It is easy to see that it factors densely, but not projectively. $\square$

Dense factorization and uniform algebras

Example Consider the 'road-runner set' X, defined by discs $\mathbb{D}(x_n, r_n)$. Then M_0 in $R(X)$ factors iff $\sum_{i=1}^{\infty} r_i/x_i = \infty$ (**Melnikov**), but factors densely iff $\sum_{i=1}^{\infty} r_i/x_i^2 = \infty$ (**Hallstrom**). Thus there are maximal ideals in algebras $R(X)$ that factor densely, but not projectively. $\square$

Side remark 1: Stu Sidney has an example of a separable uniform algebra on X and $x \in X$ such that $\{x\}$ is a one-point part, but M_x does not factor densely. $\square$

Side remark 2: There is a (non-separable) uniform algebra such that M_x factors, but $\{x\}$ is not a one-point part. $\square$

Extensions of uniform algebras - 1

Let X and Y be compact spaces, and suppose that $\Pi : Y \to X$ is a continuous surjection. Then $\Pi^* : C(X) \to C(Y)$ is defined by the formula

$$\Pi^*(f) = f \circ \Pi \quad (f \in C(X)),$$

so that Π^* is an isometric isomorphism of $C(X)$ onto a closed subalgebra of $C(Y)$. A linear contraction $T : C(Y) \to C(X)$ such that $T \circ \Pi^* = I_{C(X)}$ is an **averaging operator** for Π.

Extensions of uniform algebras - 2

Let X be a compact space, take $x_0 \in X$, and let A be a uniform algebra on X. Also suppose that (Y, y_0, B) is another uniform algebra. Then (Y, y_0, B) is an **extension** of (X, x_0, A) with respect to a continuous surjection $\Pi : Y \to X$ and an averaging operator $T : C(Y) \to C(X)$ for Π if :

(i) $\Pi^*(A) \subset B$;

(ii) $\Pi^{-1}(\{x_0\}) = \{y_0\}$;

(iii) $T(B) = A$;

(iv) $(Th)(x_0) = h(y_0)$ $(h \in C(Y))$;

(v) $|(Th)(x)| \leq |h|_{\Pi^{-1}(\{x\})}$ $(x \in X,\, h \in C(Y))$.

Extensions of uniform algebras - 3

Basic idea - this goes back to **Brian Cole** in his thesis; it is in the book of **Lee Stout**.

Start with a suitable (X, x_0, A), with $A \neq C(X)$; take an extension; keep on doing it with suitable compatibility conditions built in; index the set of extensions by the ordinals - usually up to ω_1; act sensibly at limit ordinals. We obtain an enormous, non-separable uniform algebra (Y, y_0, B), and, with care, $B \neq C(Y)$.

Cole's original construction made extensions by 'adding square roots' to obtain a uniform algebra (Y, y_0, B) with $B \neq C(Y)$ such that every element in M_{y_0} is the square of another element in M_{y_0}, and so the end point is a 'Cole algebra'. (There are also separable examples of Cole algebras.)

Examples of **Joel Feinstein** show that the final uniform algebra can have a variety of other interesting properties.

Extensions of uniform algebras - 4

Main example This is how we construct a maximal ideal M_x in a uniform algebra such that null sequences in M factor, but x is not a SBP, and so M_x does not have a BAI. The main technicality is to find an extension (Y, y_0, B) of a given uniform algebra (X, x_0, A) such that a given null sequence (f_n) in M_{x_0} factors in M_{y_0}.

We construct a compact subspace Y of the space $X \times \mathbb{C}^{\mathbb{N}} \times \mathbb{C}$ that satisfies certain conditions, namely a point $(x, (z_n), w)$ is such that:

(i) $z_n w = f_n(x) \ (n \in \mathbb{N})$;

(ii) $|w| = k_x := \max\{|f_n(x)|^{1/2} : n \in \mathbb{N}\}$;

(iii) $|z_n|^2 \leq |f_n(x)| \ (n \in \mathbb{N})$.

Extensions of uniform algebras - 5

The continuous projection is

$$\Pi : (x, (z_n), w) \mapsto x, \quad Y \to X.$$

Also we have $p_n : (x, (z_n), w) \mapsto z_n,\ Y \to \mathbb{C}$, and $q : (x, (z_n), w) \mapsto w,\ Y \to \mathbb{C}$.

When $k_x = 0$, the ‘fibre’ above x is a singleton, otherwise it is a lot of circles.

Then take B to be smallest closed subalgebra of $(C(Y), |\cdot|_Y)$ containing $\Pi^*(A)$ and all of the functions p_n and q.

The map $T : C(Y) \to C(X)$ is given by

$$(Th)(x) = \frac{1}{2\pi} \int_0^{2\pi} h\left(x, (f_n(x)/k_x)\, \mathrm{e}^{-\mathrm{i}\theta}, k_x \mathrm{e}^{\mathrm{i}\theta}\right)\, \mathrm{d}\theta$$

for $h \in C(Y)$. □

Extensions of uniform algebras - 6

Our conclusion after some more technicalities is the following:

Theorem There are a natural uniform algebra A on a compact space X and a point $x \in X$ such that all null sequences in M_x factor, but M_x does not have a BAI, equivalently, x is not a SBP for A, and, further, such that each element in M_x is the square of another element in M_x and $\{y\}$ is a one-point part with respect to A for each $y \in X$. $\square$

Again note that our example is not separable.

Esterle's classification of CRBAs

The 3rd conference on Banach algebras was held at Calstate, Long Beach, 13-31 July, 1981; the conference proceedings were published as *Lecture Notes in Mathematics*, 975. The most impressive paper in these proceedings is **Jean Esterle**'s 'Classification of CRBAs'; see my book, §4.9.

Esterle's classification has nine classes, each smaller than the one before. In each case, a class is distinct from its predecessor, save that maybe his Classes III and IV coincide, and maybe his Classes V and VI coincide.

Class V is defined to consist of the CRBAs R such that there is $a \neq 0$ in R with $a \in \overline{a^2 R}$; with CH, this condition is equivalent to 'for each infinite compact space K there is a discontinuous homomorphism from $C(K)$ into $R^\sharp$.'

Two classes

Let A be a CBA, and consider the following two properties:

(A) $\varprojlim a^n \cdot A \neq \{0\}$ for some $a \in A$;

(B) $\varprojlim a_1 \cdots a_n \cdot A \neq \{0\}$ for some (a_n) in A.

These two conditions specify classes III and IV, respectively, of Esterle in the case of CRBAs. In the case of integral domains A, we have:

(A) there exists an element $a \in A^\bullet$ that can be factored successively as $a = ba_1$, $a_1 = ba_2$, $a_2 = ba_3$, ... for some b and (a_n) in A;

(B) there exists an element $a \in A^\bullet$ that can be factored successively as $a = b_1a_1$, $a_1 = b_2a_2$, $a_2 = b_3a_3$, ... for some sequences (a_n) and (b_n) in A.

Clearly (A) $\Rightarrow$ (B) and our (IV) $\Rightarrow$(B).

Another uniform algebra - 1

We cannot distinguish (A) and (B) for CRBAs, but we can do this for uniform algebras.

Let Π be the open half-plane

$$\Pi = \{z = x + \mathrm{i}y \in \mathbb{C} : x > 0\},$$

and let $A = A^b(\overline{\Pi})$, the (non-separable) uniform algebra of all bounded, continuous functions on $\overline{\Pi}$ that are analytic on Π.

For

$$f = \sum\{\alpha_r \delta_r : r \in \mathbb{R}^+\} \in \ell^{\,1}(\mathbb{R}^+)\,,$$

denote its Laplace transform by $\mathcal{L}f$, so that

$$(\mathcal{L}f)(z) = \sum\{\alpha_r \mathrm{e}^{-zr} : r \in \mathbb{R}^+\} \quad (z \in \overline{\Pi}).$$

Denote by M and B, respectively, the closures in $(A, |\cdot|_{\overline{\Pi}})$ of $\{\mathcal{L}f : f \in \ell^{\,1}(\mathbb{Q}^{+\bullet})\}$ and $\{\mathcal{L}f : f \in \ell^{\,1}(\mathbb{Q}^+)\}$.

Thus B is a separable, unital, closed subalgebra of A, and hence a uniform algebra on $\overline{\Pi}$, and M is a maximal ideal in B.

Another uniform algebra - 2

Lemma 1 The maximal ideal M factors projectively. $\square$

Let I consist of the functions $F \in B$ such that $|F(z)| = O(\mathrm{e}^{-ax})$ as $z \to \infty$ in $\overline{\Pi}$ for some $a > 0$.

Lemma 2 Suppose that $F \in I$. Then

$$\bigcap_{n=1}^{\infty} F^n B = \{0\}.$$

$\square$

Proposition Let $F \in M \setminus I$. Then there exists $z \in \Pi$ such that $F(z) = 0$.

Proof This is an extension of a classical theorem of H. Bohr using Nevanlinna's theorem. $\square$

Another uniform algebra - 3

Theorem The maximal ideal M of the separable, unital uniform algebra B factors projectively, and so satisfies (B), but M does not satisfy (A), and hence null sequences do not factor in M.

Proof To show that M does not satisfy (A), it suffices to show that $\bigcap F^n M = \{0\}$ for each $F \in M$. If $F \in I$, this follows from Lemma 2. If $F \in M \setminus I$, by the proposition there exists $z \in \Pi$ such that $F(z) = 0$, and so each $G \in \bigcap F^n M$ is analytic on a neighbourhood of z and has a zero of infinite order at z, and so $G = 0$, giving the result in this case. $\square$

Transfer to the disc - 1

We can transfer the above algebras B and M from the half-plane Π to the unit disc $\mathbb{D}$. Indeed, take $r \in \mathbb{R}^{+\bullet}$, and define

$$f_r(z) = \exp\left(r\left(\frac{z+1}{z-1}\right)\right) \quad (z \in \overline{\mathbb{D}} \setminus \{1\})\,.$$

The functions f_r belong to $H^\infty(\mathbb{D})$. So B is the unital subalgebra of $H^\infty(\mathbb{D})$ generated by the functions f_r for $r \in \mathbb{Q}^+$.

Restrict $H^\infty(\mathbb{D})$ to the fibre above 1, and call the character space of the restriction algebra $\mathfrak{M}_1$, as in Hoffman. The common zero set of the functions f_r is called Z; it is non-empty, the union of Gleason parts for $H^\infty(\mathbb{D})$, and disjoint from the Shilov boundary.

Consider the compact space K formed by identifying the points of $\mathfrak{M}_1$ that are not separated by B.

Transfer to the disc - 2

Theorem The algebra B is a natural, separable uniform algebra on K, and the point corresponding to Z gives the maximal ideal M, and so is a one-point part off the Shilov boundary. Thus M does not have a BAI, but it does factor projectively. □

Does it factor? Does it factor weakly? Do pairs factor?