"A new approach to the L^p-theory of $-\Delta + b\cdot\nabla$, and its applications to Feller processes with general drifts"

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The problem of constructing a Feller process (a diffusion) having infinitesimal generator $-\Delta + b\cdot\nabla$, $b:\mathbb{R}^d \to \mathbb{R}^d$, $d \geqslant 3$ ("a d-Brownian motion perturbed by drift b"), has been thoroughly studied in the literature, motivated by applications, as well as the search for the maximal general class of drifts b ensuring existence of the process. With a sub-critical case $b \in L^p + L^\infty$, $p > d$, settled long time ago, this search culminated in the following two distinct classes of *critical* drifts:

– the class of form-bounded vector fields $\mathbf{F}_\delta$, i.e.

$$\|b(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \sqrt{\delta} \quad \text{for some } \lambda = \lambda_\delta > 0,$$

– the Kato class $\mathbf{K}^{d+1}_\delta$, i.e.

$$\|b(\lambda-\Delta)^{-\frac{1}{2}}\|_{1\to 1} \leqslant \delta \quad \text{for some } \lambda = \lambda_\delta > 0.$$

The vector fields in $\mathbf{F}_\delta$ and $\mathbf{K}^{d+1}_\delta$ have critical (i.e. sensitive to multiplication by a constant) singularities, at isolated points or along hypersurfaces, resp. (e.g. $\sqrt{\delta}x|x|^{-2}$ or $(|x|-1)^{-\beta}$, $\beta < 1$).

Earlier, $\mathbf{K}^{d+1}_\delta$ has been recognized as the class responsible for existence of the Gaussian bounds on the fundamental solution of $-\Delta + b\cdot\nabla$ (Yu. Semenov, Qi. Zhang, also M. Aizenman, B. Simon and others) ($\Rightarrow$ an associated Feller process). $\mathbf{F}_\delta$ ensures that that $-\Delta + b\cdot\nabla$ is dissipative in L^p, $p > \frac{2}{2-\sqrt{\delta}}$ ($\Rightarrow$ a Feller process via a Moser-type iterative procedure of Kovalenko-Yu. Semenov).

What class of drifts b is responsible for existence of a Feller process associated with $-\Delta + b\cdot\nabla$? It turns out that ultimately neither Gaussian bounds nor dissipativity in L^p is related to existence of the process: the process exists for b in the class of *weakly* form-bounded vector fields $\mathbf{F}^{\frac{1}{2}}_\delta$, i.e.

$$\||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\|_{2\to 2} \leqslant \sqrt{\delta} \quad \text{for some } \lambda = \lambda_\delta > 0.$$

$\mathbf{K}^{d+1}_\delta \subsetneq \mathbf{F}^{\frac{1}{2}}_\delta$, $\mathbf{F}_{\delta^2} \subsetneq \mathbf{F}^{\frac{1}{2}}_\delta$, and

$$b \in \mathbf{F}_{\delta_1} \text{ and } \mathsf{f} \in \mathbf{K}^{d+1}_{\delta_2} \quad \Longrightarrow \quad b + \mathsf{f} \in \mathbf{F}^{\frac{1}{2}}_\delta,\ \sqrt{\delta} = \sqrt[4]{\delta_1} + \sqrt{\delta_2},$$

i.e. for the first time b can combine different kinds of singularities, e.g. $|x|^{-1}$ and $(|x|-1)^{-\beta}$, $\beta < 1$.

The construction of the process is a consequence of the $L^p(\mathbb{R}^d)$-regularity theory of $-\Delta + b\cdot\nabla$, $p > d-1$, L^p-inequalities between operator $(\lambda-\Delta)^{\frac{1}{2}}$ and "potential" $|b|$, and "the method of constructing the resolvent".

We strengthen, in particular, the recent results of R. Bass-Z.Q. Chen [Ann. Prob., 2003] and P. Kim-R. Song [Stoc. Proc. Appl., 2014] (for $-\Delta^{\frac{\alpha}{2}} + b\cdot\nabla$) for the Kato class of measure-valued drifts.

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