COMPUTATION OF WEIGHTED CAPACITY

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Abstract. We introduce a method for computing the weighted capacity of a closed plane set. The method automatically yields upper and lower bounds for the capacity, and, for compact sets, these bounds converge to the true value of the capacity. No prior knowledge of the support of the equilibrium measure is required, and indeed the method can be used to determine this support. We discuss a number of examples in detail.

1. Introduction

In a previous article [5], two of the present authors introduced a method for computing the logarithmic capacity of plane sets. Capacities for more general kernels were subsequently treated in [3]. We now seek to extend these ideas in a different direction, to encompass so-called weighted capacities. These capacities arise in potential theory in the presence of external fields. As well as being motivated by considerations from physics, this theory has applications to the study of orthogonal polynomials. A basic reference is the book of Saff and Totik [6].

The main technique for computing capacity introduced in [5] was based on a minimax principle. For reasons to be explained below, this method does not extend to weighted capacities. However, a second technique, mentioned only informally in [5], turns out to generalize rather well to the weighted case. This technique, based on a quadratic minimization program, will be our principal method in this article. We shall give a formal development of it, and use it to analyze a number of examples.

The rest of the paper is organized as follows. In §2 we summarize some basic elements of weighted potential theory, and in §3 we set up the notation for the quadratic minimization program that we shall use. The core of the paper is in §4 and §5, where we establish the upper and lower bounds for weighted capacity in terms of the quadratic minimization program, along with the corresponding convergence theorems. In §6, we present a number of illustrative examples, and we conclude in §7 by comparing our methods with other techniques in the literature. At the end of the paper there is an appendix containing some analytic computations of weighted capacity in certain special cases, which are used to test our algorithms.

2. Background on weighted potential theory

Here we briefly describe the set-up for the rest of the paper. For full details we refer to [6].

Let Σ be a closed subset of $\mathbb{C}$. A *weight* on Σ is an upper semicontinuous function $w : \Sigma \to [0, \infty)$. If Σ is unbounded, we also impose the condition that $|z|w(z) \to 0$ as $|z| \to \infty$. We write $Q_w := \log(1/w)$.

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