PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY
Received by the editor: June 23, 2008
Accepted for publication: October 26, 2008

ESSENTIALLY SPECTRALLY BOUNDED LINEAR MAPS

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(Communicated by Nigel Kalton)

ABSTRACT. Let $\mathscr{L}(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H. We characterize essentially spectrally bounded linear maps from $\mathscr{L}(H)$ onto itself. As consequences, we characterize linear maps from $\mathscr{L}(H)$ onto itself that compress different essential spectral sets such as the the essential spectrum, the (left, right) essential spectrum, and the semi-Fredholm spectrum.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Throughout this note, $\mathscr{H}$ will denote an infinite dimensional complex Hilbert space and $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded linear operators on $\mathscr{H}$. The closed ideal of all compact operators on $\mathscr{H}$ is denoted by $\mathscr{K}(\mathscr{H})$, and the Calkin algebra is denoted, as usual, by $\mathscr{C}(\mathscr{H}) := \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$. For an operator $T \in \mathscr{L}(\mathscr{H})$, let $\sigma_e(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$, and $\sigma_{SF}(T)$ denote the essential spectrum, the left essential spectrum, the right essential spectrum, and the semi-Fredholm spectrum, respectively, of T; see for instance [1]. The essential norm of the operator T is given by $\|T\|_e := \operatorname{dist}(T, \mathscr{K}(\mathscr{H}))$ and the essential spectral radius, denoted by $r_e(T)$, is the limit of the convergent sequence $(\|T^n\|_e{}^{1/n})_{n\geq 1}$. It coincides with $r(\pi(T))$ the classical spectral radius of $\pi(T)$, where π denotes the canonical quotient map from $\mathscr{L}(\mathscr{H})$ onto $\mathscr{C}(\mathscr{H})$.

New contributions to the study of linear preserver problems in $\mathscr{L}(\mathscr{H})$ have recently been made in [4, 5, 8, 20, 21, 22]. In [22], Mbekhta has treated the problem of characterizing surjective linear maps on $\mathscr{L}(\mathscr{H})$ preserving the set of Fredholm operators in both directions. He proved, in particular, that a surjective linear map on $\mathscr{L}(\mathscr{H})$ preserves the set of Fredholm operators in both directions if and only if it leaves invariant the closed ideal $\mathscr{K}(\mathscr{H})$ of all compact operators, and the induced map on the Calkin algebra, $\mathscr{C}(\mathscr{H}) := \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, is either an automorphism or an anti-automorphism

2000 *Mathematics Subject Classification.* Primary 47B49; Secondary 47A10, 47D25.

Key words and phrases. Linear preserver problem, spectrally bounded linear map, essential spectral radius, Fredholm and semi-Fredholm operators.

The second author is supported by the adjunct professorship at Laval University. The authors thank very much Professor M. Brešar for his comments and remarks which lead to the final version of this note. They also thank the organizers of "*the 5th Linear Algebra Workshop, Kranjska Gora, Slovenia (May 27 - June 5, 2008)*" and "*the 2nd International Conference On Linear Preservers and Operator Theory, Meknès, Morocco (June 2-6, 2008)*" where the main result of this note has been announced.