

# LINEAR MAPS PRESERVING THE MINIMUM AND REDUCED MINIMUM MODULI

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ABSTRACT. We describe linear maps from a  $C^*$ -algebra onto another one preserving different spectral quantities such as the minimum modulus, the surjectivity modulus, and the reduced minimum modulus.

## 1. INTRODUCTION

In the last few decades, there has been a considerable interest in the so-called linear preserver problems which concern the characterization of linear or additive maps on matrix algebras or operator algebras or more generally on Banach algebras that leave invariant a certain function, a certain subset, or a certain relation; see for instance the survey papers [8, 16, 21, 22, 29] and the references therein. One of the most important question in this active research area is the famous problem initiated by Kaplansky's discussion in [20] that asks whether a spectrum-preserving linear map from a semisimple unital Banach algebra  $\mathcal{A}$  onto another one  $\mathcal{B}$  is a Jordan isomorphism. This problem remains far from being solved even when  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -algebras but it has a positive solution when  $\mathcal{B}$  is commutative or when  $\mathcal{A}$  and  $\mathcal{B}$  are either von Neumann algebras or Banach algebras with big socles; see for instance [1, 2, 6, 7, 11, 15, 17, 19, 23, 32, 34].

Recently, Mbekhta described unital surjective linear maps on the algebra  $\mathcal{L}(H)$  of all bounded linear operators on an infinite dimensional complex Hilbert space  $H$  preserving several spectral quantities such as the minimum, the surjectivity and the reduced minimum moduli. In [25], he proved that a unital surjective linear map from  $\mathcal{L}(H)$  onto itself preserves either the minimum modulus or the surjectivity modulus if and only if it is a selfadjoint automorphism. While in [26], he showed that a unital surjective linear map on  $\mathcal{L}(H)$  preserves the reduced minimum modulus precisely when it is either a selfadjoint automorphism or a selfadjoint anti-automorphism. These results have been extended by the first two authors to the setting of surjective linear maps between  $C^*$ -algebras of real rank zero preserving different spectral quantities; see [4, 5].

For the nonunital case, Mbekhta closed his paper [26] with the following natural conjecture.

*Conjecture* ([26]). A surjective linear map  $\Phi : \mathcal{L}(H) \rightarrow \mathcal{L}(H)$  preserves the reduced minimum modulus if and only if there are unitary operators  $U, V \in \mathcal{L}(H)$

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