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On Hyperbolic Divided Differences and the Nevanlinna–Pick Problem

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Abstract. Starting from the notions of the complex pseudo-hyperbolic distance and the hyperbolic difference quotient introduced by A. F. Beardon and D. Minda in [1], we define hyperbolic divided differences for unimodularly bounded holomorphic functions in the complex unit disk and investigate their mapping properties. In particular we show that they operate on Blaschke products in the same way as the ordinary divided differences act on polynomials. As a simple corollary we obtain a multi-point Schwarz–Pick Lemma.

Using these concepts we investigate the classical interpolation problem of Pick and Nevanlinna and reformulate the Nevanlinna–Schur algorithm in terms of hyperbolic divided differences. This leads to a scheme that (formally) coincides with Newton's algorithm for polynomial interpolation.

Keywords. Hyperbolic geometry, divided differences, Nevanlinna–Pick interpolation, Newton interpolation, Schwarz Lemma.

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1. Introduction

In their paper [1] Beardon and Minda studied the class $\mathcal{H}$ of holomorphic mappings from the complex unit disk $\mathbb{D}$ into itself using ideas from hyperbolic geometry. In particular they introduced a hyperbolic difference quotient and the hyperbolic derivative of functions in $\mathcal{H}$ and pointed out analogies between polynomials in the Euclidean geometry and Blaschke products in the hyperbolic geometry.

Here we extend the definitions of Beardon and Minda to include also the complex unit circle $\mathbb{T}$ as the "horizon" of the hyperbolic plane $\mathbb{D}$, and an abstract point ∞ which stands for "everything beyond the horizon". In this context we use the notations

$$\overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{T}, \quad \widehat{\mathbb{D}} := \overline{\mathbb{D}} \cup \{\infty\}.$$

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