Abstract

It has been known for some time that $l^1(\mathbf{Z}+)$, the unital semigroup algebra of $\mathbf{N}$, is not weakly amenable, that is $\mathcal{H}^1(l^1(\mathbf{Z}+), l^1(\mathbf{Z}+)') \neq 0$. This may lead one to believe that $\mathcal{H}^n(l^1(\mathbf{Z}+), l^1(\mathbf{Z}+)')$, the higher simplicial cohomology groups, are also non-zero for $n \geq 2$. However, Johnson showed that the alternating cohomology of this algebra vanishes in all dimensions strictly greater than 1. Then, in a systematic calculation of second cohomology groups, Dales and Duncan showed that the second simplicial cohomology of $l^1(\mathbf{Z}+)$ is trivial. This leads to the conjecture that all the simplicial cohomology groups of $l^1(\mathbf{Z}+)$ vanish for $n \geq 2$.

In this paper, we show that the third simplicial cohomology group of $l^1(\mathbf{Z}+)$ vanishes. We first use the Connes-Tzygan exact sequence to prove that this is equivalent to the vanishing of the third cyclic cohomology group $\mathcal{H}C^3(\mathcal{I}, \mathcal{I}')$, where $\mathcal{I}$ is the non-unital Banach algebra $l^1(\mathbf{N})$, and then prove that $\mathcal{H}C^3(\mathcal{I}, \mathcal{I}') = 0$.