B. J. Cole and T. J. Ransford, **Subharmonicity without upper semicontinuity**, *J. Funct. Anal.*, 147 (1997), 420–442.

Abstract
Let Ω be an open subset of $\mathbf{R}^d$ ($d \geq 2$). Given $x \in \Omega$, a *Jensen measure* for x is a Borel probability measure μ, supported on a compact subset of Ω, such that each subharmonic function u on Ω satisfies

$$u(x) \leq \int u \, d\mu. \qquad (*)$$

A function u on Ω (subject to certain measurability conditions, much weaker for example than upper semicontinuity) is called *quasi-subharmonic* if it satisfies $(*)$ for each $x \in \Omega$ and each Jensen measure μ for x.
Our first result is that u is quasi-subharmonic on Ω if and only if its upper semicontinuous regularization u^* is subharmonic on Ω and equal to u outside a set of capacity zero. This easily implies, for example, Cartan's classical theorem on the supremum of a locally bounded family of subharmonic functions.
Our second result is a converse to Cartan's theorem: provided that Ω has a Green's function, every quasi-subharmonic function on Ω *is* the supremum of some family of subharmonic functions.
These ideas eventually lead to the following duality theorem. If ϕ is a Borel function locally bounded above on Ω, and if Ω has a Green's function, then for each $x \in \Omega$,

$$\sup\Big\{ v(x) : v \text{ subharmonic on } \Omega,\ v \leq \phi \Big\} = \inf\Big\{ \int \phi \, d\mu : \mu \text{ is a Jensen measure for } x \Big\}.$$

We also investigate to what extent these results carry over to plurisubharmonic functions.